Numerical Analysis of DiffEq HW 1

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1.1 We first need to establish a lemma:

Lemma 1. *$hf(t_{n+\frac{1}{2}}, \frac{1}{2}(y(t_n) + y(t_{n+1})) = hf(t_{n+\frac{1}{2}}, y(t_{n+\frac{1}{2}})) + \eta$ where η is $\mathcal{O}(h^3)$.*

$$\begin{aligned} ||\eta|| = h||f(t_{n+\frac{1}{2}}, \frac{1}{2}(y(t_n) + y(t_{n+1})) - f(t_{n+\frac{1}{2}}, y(t_{n+\frac{1}{2}})|| \\ \leq h\lambda||\frac{1}{2}(y(t_n) + y(t_{n+1})) = y(t_{n+\frac{1}{2}}) \\ \leq h\frac{\lambda}{2}|y(t_n) + y(t_n + 1) - 2y(t_{n+\frac{1}{2}})|| \end{aligned}$$

Then Taylor expansion around $y(t_n)$ gives us:

$$\begin{aligned} ||\eta|| &\leq \frac{h\lambda}{2}||y + (y + hy' - 2y(y + \frac{1}{2}hy') + \mathcal{O}(h^2)|| \\ &= \frac{h\lambda}{2}||y + y + hy^2 - 2y^2 - hyy' + \mathcal{O}(h^2)|| \\ &= \mathcal{O}(h^3) \end{aligned}$$

Proving the convergence of the implicit midpoint rule:
The implicit midpoint rule is:

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, \frac{1}{2}(y_n + y_{n+1})). \tag{1}$$

Substituting the exact value then gives us:

$$y(t_{n+1}) = y(t_n) + hf(t_n + \frac{h}{2}, \frac{1}{2}(y(t_n) + y(t_{n+1})) + \mathcal{O}(h^2) \tag{2}$$

Following closely the proof of (1.9) and (1.4) we subtract (2) from (1), and applying the lemma to get:

$$e_{n+1} = e_n + h(f(t_{n+\frac{1}{2}}, \frac{1}{2}(y_n + y_{n+1})) - f(t_{n+\frac{1}{2}}, \frac{1}{2}(y(t_n) + y(t_{n+\frac{1}{2}})) + \mathcal{O}(h^3)$$

$$||e_{n+1}|| \leq ||e_n|| + \frac{\lambda h}{2}(||e_{n+1}|| + ||e_n||) + \mathcal{O}(h^3)$$

We then reproduce the steps in Iserles:

$$||e_{n+1}|| - \frac{\lambda h}{2}||e_{n-1}|| \leq ||e_n|| + \frac{\lambda h}{2}||e_n|| + \mathcal{O}(h^3)$$
$$||e_{n+1}|| \leq \left(\frac{1+\frac{\lambda h}{2}}{1-\frac{\lambda h}{2}}\right)||e_n|| + \left(\frac{c}{1+\frac{\lambda}{2}}\right) + \mathcal{O}(h^3)$$

And similar to Iserles we claim that:

$$||e_n|| \leq \frac{c}{\lambda}\left[\left(\frac{1+\frac{\lambda h}{2}}{1-\frac{\lambda h}{2}}\right)^n - 1\right]h^2$$

We will use induction to prove this step. Clearly it holds for $n = 0$. Assume that the above inequality holds upto and including $n \in \mathbb{N}$, then for $n+1$ we have:

$$||e_{n+1}|| \leq \left(\frac{1+\frac{\lambda h}{2}}{1-\frac{\lambda h}{2}}\right)\frac{c}{\lambda}\left[\left(\frac{1+\frac{\lambda h}{2}}{1-\frac{\lambda h}{2}}\right)^n - 1\right]h^2 + \left(\frac{c}{1+\frac{\lambda}{2}}\right) + \mathcal{O}(h^3)$$
$$= \frac{c}{\lambda}\left[\left(\frac{1+\frac{\lambda h}{2}}{1-\frac{\lambda h}{2}}\right)^{n+1}\right]h^2 - \frac{c}{\lambda}\left(\frac{1+\frac{\lambda h}{2}}{1-\frac{\lambda h}{2}}\right)h^2 + \mathcal{O}(h^3)$$
$$= ||e_n|| \leq \frac{c}{\lambda}\left[\left(\frac{1+\frac{\lambda h}{2}}{1-\frac{\lambda h}{2}}\right)^{n+1} - 1\right]h^2$$

To show that the θ method is convergent, we define the theta method as:

$$y_{n+1} = y_n + h[\theta f(t_n, y_n) + (1-\theta)f(t_{n+1}, y_{n+1})]$$

Whereas substituting exact solutions gives us:

$$y(t_{n+1})y(t_n) + h[\theta f(t_n, y(t_n)) + (1-\theta)f(t_{n+1}, y(t_{n+1}))] + \mathcal{O}(h^3)$$

Once again subtracting (3) from (4) gives us:

$$e_{n+1} = e_n + h[\theta f(t_n, y_n) - \theta f(t_n, y(t_n)) + (1-\theta)f(t_{n+1}, y(t_{n+1})) + (1-\theta)f(t_{n+1}, y(t_{n+1}))] + \mathcal{O}(h^3)$$
$$||e_{n+1}|| \leq ||e_n|| + h[\theta\lambda||e_n|| + (1-\theta)\lambda||e_{n+1}||] + \mathcal{O}(h^3)$$
$$||e_{n+1}|| \leq \left(\frac{1+h\theta\lambda}{1-h(1-\theta)\lambda}\right)||e_n|| + \frac{c}{1-h(1-\theta)\lambda} + ch^3 \quad \text{for some } c$$

Similar to the trapezoid method we will argue:

$$||e_n|| \leq \frac{c}{\lambda}\left[\left(\frac{1+h\theta\lambda}{1-h(1-\theta)\lambda}\right)^n - 1\right]h^2$$

We argue via induction. Clearly we have that it is true for $n = 0$ as at that point the exact and approximate solutions are the same. Assume now that it is true upto and including n. We need to prove for $n + 1$:

$$||e_{n+1}|| \leq \left(\frac{1 + h\theta\lambda}{1 - h(1-\theta)\lambda}\right)||e_n|| + \frac{c}{1 - h(1-\theta)\lambda} + ch^3$$
$$||e_{n+1}|| \leq \left(\frac{1 + h\theta\lambda}{1 - h(1-\theta)\lambda}\right)\left(\frac{c}{\lambda}\left[\left(\frac{1 + h\theta\lambda}{1 - h(1-\theta)\lambda}\right)^n - 1\right]h^2\right) + \frac{c}{1 - h(1-\theta)\lambda} + ch^3$$

Now observe that θ varies between 1 and 0. Thus $\left(\frac{1+h\theta\lambda}{1-h(1-\theta)\lambda}\right)$ varies between $1 + h\lambda$ and $\frac{1}{1-h\lambda}$, both of which are bigger than one. As such we can continue and say:

$$||e_n|| \leq \frac{c}{\lambda}\left[\left(\frac{1 + h\theta\lambda}{1 - h(1-\theta)\lambda}\right)^{n+1} - 1\right]h^2$$

1.2a Let $y' = Ay$, and let $e_n = y_n - y(nh)$ We want to prove using induction:

$$||e_n||_2 \leq ||y_0||_2 \max_{\lambda \in \sigma(A)} |(1 - h\lambda)^n - e^{nh\lambda}|$$

But before that we make an observation:

$$e_n = y_n - y(nh)$$

And since we are using Euler method, we can say:

$$y(t_{n+1}) = y(t_n) + hy'(t) + \mathcal{O}(h^2)$$
$$y_{n+1} - y(t_{n-1}) = y_n - y(nh) + h[(f(t_n, y(t_n)) - f(t_n, y_n))]$$

Substitution A gives us:

$$e_{n+1} = e_n + h[Ay_n - Ay(nh)] + \mathcal{O}(h^2)$$
$$||e_{n+1}|| \leq ||e_n||_2 + h\lambda||e_n||_2 + \mathcal{O}(h^2)$$
$$||e_{n+1}|| \leq ||e_n||_2(1 + h\lambda) + \mathcal{O}(h^2)$$

For the induction part we observe that the statement clearly holds true for $n = 0$ since then we get:

$$||e_0||_2 \leq ||y_0||_2 \max_{\lambda \in \sigma(A)} |(1 + h\lambda)^0 - e^0|$$
$$0 \leq 0$$

Using it as our base case, assume the inequality holds for upto and including n the for $||e_{n+1}||$ we have:

$$
\begin{aligned}
||e_{n+1}||_2 &\leq ||e_n||_2(1+h\lambda) + \mathcal{O}(h^2) \\
||e_{n+1}||_2 &\leq ||y_0||_2 \max_{\lambda\in\sigma(A)} |(1+h\lambda)^n - e^{nh\lambda}|(1+h\lambda) + \mathcal{O}(h^2 x) \\
&\leq ||y_0||_2 \max_{\lambda\in\sigma(A)} |(1+h\lambda)^{n+1} - [e^{nh\lambda} + h\lambda e^{nh\lambda}]| \\
&\leq ||y_0||_2 \max_{\lambda\in\sigma(A)} |(1+h\lambda)^{n+1} - e^{(n+1)h\lambda}|
\end{aligned}
$$

1.2b From the hint, we first seek to prove $1 + x \leq e^x$. Let $f(x) = e^x - x - 1$, then $f'(x) = e^x - 1$ and $f''(x) = e^x$. There is a global minimum of 0 and this function is concave up, and so $f(x) > 0$ over all x and hence $e^x \geq 1 + x$.
Following the hint again we seek to prove that $1 + x + \frac{x^2}{2} \geq e^x$. Observe the series expansion of e^x is $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\mathcal{O}(x^3)$. Thus:

$$
\begin{aligned}
1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \mathcal{O}(h^4) - 1 - x - \frac{x^2}{2} \\
\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} + \frac{x^3}{3!} + \mathcal{O}(h^4) - \cancel{1} - \cancel{x} - \cancel{\frac{x^2}{2}} \\
\frac{x^3}{3!} + \mathcal{O}(h^4) < 0 \text{ as } x \in [-1, 0]
\end{aligned}
$$

We use a similar logic for the last part of the hint. Observe that: $(a - b)^n = \Sigma_{i=0}^n \binom{n}{i} a^{n-i} b^i = a^n - na^{n-1}b + ...$ So:

$$
\begin{aligned}
&\Sigma_{i=1}^n \binom{n}{i} a^{n-i} b^i - a^n + na^{n-1}b \\
&\Sigma_{i=2}^n \binom{n}{i} a^{n-i} b^i \geq 0 \text{ as a is close to being 1 and b is small}
\end{aligned}
$$

For the actual proof, let $a = e^x$ and $b = \frac{1}{2}x^2$. We then get:

$$
e^{nx} - \frac{1}{2}nx^2 e^{(n-1)x} \leq (e^x - \frac{x^2}{2})^n \leq (1+x)^n \leq e^{nx}
$$

1.4