Reformulation without f

Shakil Rafi

May 25, 2023

Lemma 0.0.1. *Let* $\Theta = \bigcup_{n\in\mathbb{N}}$, $d, m, \mathfrak{d} \in \mathbb{N}$, $T \in (0,\infty)$, $\mathrm{Act} \in C(\mathbb{R},\mathbb{R})$, $\mathfrak{I}, \mathbf{F}, \mathbf{G} \in \mathrm{NN}$ *satisfy* $\mathrm{Lay}(\mathfrak{I}) = (1,\mathfrak{d},1)$, $\mathrm{Rlz}_{\mathrm{Act}}(\mathfrak{I}) = \mathbb{I}_1$, $\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{F}) \in C(\mathbb{R},\mathbb{R})$, *and* $\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{G}) \in C(\mathbb{R}^d,\mathbb{R})$, *for every* $\theta \in \Theta$ *let* $\mathcal{U}^\theta : [0,T] \to [0,T]$ *and* $\mathcal{W}^\theta : [0,T] \to \mathbb{R}^d$, *be functions, for every* $\theta \in \Theta$, $n \in \mathbb{N}_0$ *let* $U^\theta_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *satisfy for all* $t \in [0,T]$, $x \in \mathbb{R}^d$, *that:*

$$U^\theta_n(t,x) = \frac{\mathbb{1}_{\mathbb{N}}(n)}{M^n}\left[\sum_{k=1}^{m^n}\left(\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{G})\right)\left(x+\mathcal{W}^{(\theta,0,-k)}_{T-t}\right)\right]$$
$$+\sum_{i=1}^{n-1}\frac{T-t}{M^{n-i}}\left[\sum_{k=1}^{M^{n-i}}\left(\left(\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{F})\circ U^{(\theta,i,k)}_i\right)-\mathbb{1}_{\mathbb{N}}(i)\left(\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{F})\circ U^{(\theta,-i,k)}\right)\right)\left(\mathcal{U}^{(\theta,i,k)}_t, x+\mathcal{W}^{(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t-t}\right)\right]$$

and let $\mathbf{U}^\theta_{n,t} \in \mathrm{NN}$, $t \in [0,T]$, $n \in \mathbb{Z}$, $\theta \in \Theta$, *satisfy for all* $\theta \in \Theta$, $n \in \mathbb{N}$, $t \in [0,T]$ *that* $\mathbf{U}^\theta_{0,t} = ((0\,0\,\cdots\,0),0) \in \mathbb{R}^{1\times d}\times\mathbb{R}$ *and:*

$$\begin{aligned}\mathbf{U}^\theta_{n,t} = &\left[\oplus_{k=1}^{M^n}\left(\frac{1}{M^n}\circledast\left(\mathbf{G}\bullet\mathrm{Aff}_{\mathbb{I}_d,\mathcal{W}^{(\theta,0,-k)}_{T-t}}\right)\right)\right]\\ &\boxplus_{\mathfrak{I}}\left[\boxplus_{i=0,\mathfrak{I}}\left[\left(\frac{T-t}{M^{n-i}}\right)\circledast\left(\boxplus_{k=1,\mathfrak{I}}^{M^{n-i}}\left(\left(\mathbf{F}\bullet\mathbf{U}^{(\theta,i,k)}_{i,\mathcal{U}^{(\theta,i,k)}_t}\right)\bullet\mathrm{Aff}_{\mathbb{I}_d,\mathcal{W}^{(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}-t}}\right)\right)\right]\right]\\ &\boxplus_{\mathfrak{I}}\left[\boxplus_{i=0,\mathfrak{I}}^{n-1}\left[\left(\frac{(t-T)\mathbb{1}_{\mathbb{N}}(i)}{M^{n-i}}\right)\circledast\left(\boxplus_{k=1,\mathfrak{I}}^{M^{n-i}}\left(\left(\mathbf{F}\bullet\mathbf{U}^{(\theta,-i,k)}_{\max\{i-1,0\},\mathcal{U}^{(\theta,i,k)}_t}\right)\bullet\mathrm{Aff}_{\mathbb{I}_d,\mathcal{W}^{(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t-t}}\right)\right)\right]\right]\end{aligned}$$

Theorem 0.0.2. *Let* $p, q, r, L, C, \alpha_0, \alpha_1, \beta_0, \beta_1, T \in [0,\infty)$, $\mathfrak{q} \in [2,\infty)$, $\mathrm{Act} \in C(\mathbb{R},\mathbb{R})$, $\mathfrak{I} \in \mathrm{NN}$. $(\mathbf{F}_{d,\varepsilon})_{(d,\varepsilon)\in\mathbb{N}_0\times(0,1]} \subsetneq \mathrm{NN}$. *For every* $d \in \mathbb{N}_0$ *let* $f_d \in C\left(\mathbb{R}^{\max\{d,1\}},\mathbb{R}\right)$, *for every* $d \in \mathbb{N}$ *let* $\nu_d : \mathcal{B}(\mathbb{R}^d) \to [0,1]$ *be a probability measure, and assume for all* $d \in \mathbb{N}_0$, $v, w \in \mathbb{R}$, $x \in \mathbb{R}^{\max\{d,1\}}$, $\varepsilon \in (0,1]$ *that* $\left(\int_{\mathbb{R}^d}\|y\|^{pq\mathfrak{q}}\,\nu_d(dy)\right)^{\frac{1}{pq\mathfrak{q}}} \leqslant Cd^r$, $\mathrm{Hid}(\mathfrak{I}) = 1$, $\mathrm{Rlz}_{\mathrm{Act}}(\mathfrak{I}) = \mathrm{Id}_{\mathbb{R}}$, $\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{F}_{d,\varepsilon}) \in C\left(\mathbb{R}^{\max\{d,1\}},\mathbb{R}\right)$, $\max\{|f_0(v)-f_0(w)|, |(\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{F}_{0,\varepsilon}))(x)-(\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{F}_{0,\varepsilon}))(x)|\} \leqslant L|v-w|$, $\varepsilon^{\alpha_{\min\{d,1\}}}\,\mathrm{Dep}(\mathbf{F})_{d,\varepsilon} + \varepsilon^{\beta_{\min\{d,1\}}}\,\|\mathrm{Lay}(\mathbf{F}_{d,\varepsilon})\|_{\max} \leqslant C(\max\{x,1\}^p)$, *and:*

$$\varepsilon\,|(\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{F}_{d,\varepsilon}))(x)| + |f_d(x)-(\mathrm{Rlz}_{\mathrm{Act}}(\mathbf{F}_{d,\varepsilon}))(x)| \leqslant \varepsilon C(\max\{x,1\})^p(1+\|x\|)^{pq} \qquad (0.0.1)$$

It is then the case that for every $d \in \mathbb{N}$, *there exists a* $u_d \in C\left([0,T]\in\mathbb{R}^d,\mathbb{R}\right)$ *with the following properties:*

(i) u_d *is polynomially growing.*

(ii) u_d *is a viscosity solution.*

(iii) u_d *is a solution to:*

$$\left(\frac{\partial}{\partial t}u_d\right)(t,x)+\frac{1}{2}\operatorname{Trace}\left(\sigma_d\left(x\right)\left[\sigma_d\left(x\right)\right]^*\left(\operatorname{Hess}_x u_d\right)(t,x)\right)+\left\langle u_d\left(x\right),\left(\nabla_x u_d\right)(t,x)\right\rangle+\alpha_d(x)u_d(t,x)=0$$

with $u_d\left(T,x\right)=g_d\left(x\right)$ *for* $(t,x)\in(0,T)\times\mathbb{R}^d$, *and*

(iv) there exist $\left(\mathbf{U}_{d,t,\varepsilon}\right)_{(d,t,\varepsilon)\in\mathbb{N}\times[0,T]\times(0,1]}$ *and* $\eta\in\left(\eta_\delta\right)_{\delta\in(0,\infty)}:\mathbb{R}\to\mathbb{R}$ *such that for all* $d\in\mathbb{N}$, $t\in[0,T]$, $\varepsilon\in(0,1]$, $\delta\in(0,\infty)$ *it holds that* $\operatorname{Rlz}_{\operatorname{Act}}\left(\mathbf{U}_{d,t,\varepsilon}\right)\in C\left(\mathbb{R}^d,\mathbb{R}\right)$, $\operatorname{Param}\left(\mathbf{U}_{d,t,\varepsilon}\right)\leqslant\eta_\delta d^{p(7+4q+(2+q)\delta)}\varepsilon^{-(4+2\delta+\max\{\alpha_0,\alpha_1\}+2\max\{\beta_0,\beta_1\})}$ *and:*

$$\left(\int_{\mathbb{R}^d}\left|u_d\left(t,x\right)-\left(\operatorname{Rlz}_{\operatorname{Act}}\left(\mathbf{U}_{d,t,\varepsilon}\right)\right)(x)\right|^{\mathfrak{q}}\nu_d\left(dx\right)\right)^{\frac{1}{\mathfrak{q}}}\leqslant\varepsilon \tag{0.0.2}$$

Proof. The proof of Theorem 0.0.2 is thus complete ☐