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Abstract

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Lemma 2.1. *Let $p \in (2, \infty)$, $n \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_i \colon \Omega \to \mathbb{R}$, $i \in \{1, 2, \ldots, n\}$, be i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Then it holds that*

$$\left(\mathbb{E}\Big[\big|\mathbb{E}[X_1] - \tfrac{1}{n}\big(\textstyle\sum_{i=1}^n X_i\big)\big|^p\Big]\right)^{1/p} \leq \left[\tfrac{p-1}{n}\right]^{1/2}\left(\mathbb{E}\Big[\big|X_1 - \mathbb{E}[X_1]\big|^p\Big]\right)^{1/p}. \tag{2.1}$$

Proof of Lemma 2.1. First, observe that the hypothesis that for all $i \in \{1, 2, \ldots, n\}$ it holds that $X_i \colon \Omega \to \mathbb{R}$ are i.i.d. random variables ensures that

$$\begin{aligned} \mathbb{E}\Big[\big|\mathbb{E}[X_1] - \tfrac{1}{n}\big(\textstyle\sum_{i=1}^n X_i\big)\big|^p\Big] &= \mathbb{E}\Big[\big|\tfrac{1}{n}\big(\textstyle\sum_{i=1}^n (\mathbb{E}[X_1] - X_i)\big)\big|^p\Big] \\ &= n^{-p}\,\mathbb{E}\Big[\big|\textstyle\sum_{i=1}^n (\mathbb{E}[X_i] - X_i)\big|^p\Big]. \end{aligned} \tag{2.2}$$

This, the fact that for all $i \in \{1,2,\dots,n\}$ it holds that $X_i\colon \Omega \to \mathbb{R}$ are i.i.d. random variables, and, e.g., Rio [2, Theorem 2.1] (applied with $p \curvearrowleft p$, $(S_i)_{i\in\{0,1,\dots,n\}} \curvearrowleft (\sum_{k=1}^{i}(\mathbb{E}[X_k] - X_k))_{i\in\{0,1,\dots,n\}}$, $(X_i)_{i\in\{1,2,\dots,n\}} \curvearrowleft (\mathbb{E}[X_i] - X_i)_{i\in\{1,2,\dots,n\}}$ in the notation of Rio [2, Theorem 2.1]) assure that

$$\begin{aligned}\Big(\mathbb{E}\Big[\big|\mathbb{E}[X_1] - \tfrac{1}{n}\big(\textstyle\sum_{i=1}^{n} X_i\big)\big|^p\Big]\Big)^{2/p} &= \tfrac{1}{n^2}\Big(\mathbb{E}\Big[\big|\textstyle\sum_{i=1}^{n}(\mathbb{E}[X_i] - X_i)\big|^p\Big]\Big)^{2/p} \\ &\le \tfrac{(p-1)}{n^2}\Big[\textstyle\sum_{i=1}^{n}\Big(\mathbb{E}\Big[\big|\mathbb{E}[X_i] - X_i\big|^p\Big]\Big)^{2/p}\Big] \\ &= \tfrac{(p-1)}{n^2}\Big[n\Big(\mathbb{E}\Big[\big|\mathbb{E}[X_1] - X_1\big|^p\Big]\Big)^{2/p}\Big] \\ &= \tfrac{(p-1)}{n}\Big(\mathbb{E}\Big[\big|\mathbb{E}[X_1] - X_1\big|^p\Big]\Big)^{2/p}.\end{aligned} \tag{2.3}$$

The proof of Lemma 2.1 is thus complete. □

Corollary 2.2. *Let $p \in [2,\infty)$, $n \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_i\colon \Omega \to \mathbb{R}$, $i \in \{1,2,\dots,n\}$, be i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Then it holds that*

$$\Big(\mathbb{E}\Big[\big|\mathbb{E}[X_1] - \tfrac{1}{n}(\textstyle\sum_{i=1}^{n} X_i)\big|^p\Big]\Big)^{1/p} \le \big[\tfrac{p-1}{n}\big]^{1/2}\Big(\mathbb{E}\Big[\big|X_1 - \mathbb{E}[X_1]\big|^p\Big]\Big)^{1/p}. \tag{2.4}$$

Proof of Corollary 2.2. Note that, e.g., Grohs et al. [1, Lemma 2.3] and Lemma 2.1 prove (2.4). The proof of Corollary 2.2 is thus complete. □

Definition 2.3. Let $p \in [2,\infty)$. Then we denote by $\mathfrak{K}_p \in \mathbb{R}$ the real number given by

$$\mathfrak{K}_p = \inf\left\{ c \in \mathbb{R}\colon \begin{bmatrix}\text{It holds for every probability space } (\Omega,\mathcal{F},\mathbb{P}) \text{ and every} \\ \text{random variable } X\colon \Omega \to \mathbb{R} \text{ with } \mathbb{E}[|X|] < \infty \text{ that} \\ \big(\mathbb{E}\big[|X - \mathbb{E}[X]|^p\big]\big)^{1/p} \le c\big(\mathbb{E}\big[|X|^p\big]\big)^{1/p}\end{bmatrix}\right\}. \tag{2.5}$$

Corollary 2.4. *Let $p \in [2,\infty)$, $n \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_i\colon \Omega \to \mathbb{R}$, $i \in \{1,2,\dots,n\}$, be i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Then*

$$\Big(\mathbb{E}\Big[\big|\mathbb{E}[X_1] - \tfrac{1}{n}(\textstyle\sum_{i=1}^{n} X_i)\big|^p\Big]\Big)^{1/p} \le \tfrac{\mathfrak{K}_p\sqrt{p-1}}{n^{1/2}}\Big(\mathbb{E}\Big[\big|X_1\big|^p\Big]\Big)^{1/p} \tag{2.6}$$

(cf. Definition 2.3).

Proof of Corollary 2.4. Observe that Definition 2.3 and Corollary 2.2 show that (2.6) holds. The proof of Corollary 2.4 is thus complete. □

References

[1] Grohs, P., Hornung, F., Jentzen, A., and Von Wurstemberger, P. A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of Black-Scholes partial differential equations. *arXiv:1809.02362* (2018), 124 pages. *To appear in Mem. Amer. Math. Soc.*

[2] Rio, E. Moment inequalities for sums of dependent random variables under projective conditions. *J. Theor. Probab. 22*, 1 (2009), 146–163.