

On existence and uniqueness properties for solutions of stochastic fixed point equations

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August 12, 2019

Abstract

The Feynman–Kac formula implies that every suitable classical solution of a semilinear Kolmogorov partial differential equation (PDE) is also a solution of a certain stochastic fixed point equation (SFPE). In this article we study such and related SFPEs. In particular, the main result of this work proves existence of unique solutions of certain SFPEs in a general setting. As an application of this main result we establish the existence of unique solutions of SFPEs associated with semilinear Kolmogorov PDEs with Lipschitz continuous nonlinearities even in the case where the associated semilinear Kolmogorov PDE does not possess a classical solution.

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1 Introduction

The Feynman–Kac formula implies that every suitable classical solution of a semilinear Kolmogorov partial differential equation (PDE) is also a solution of a certain stochastic fixed point equation (SFPE). In this article we study such and related SFPEs. The main result of this article, Theorem 2.9 in Section 2.5 below, shows the existence of unique solutions of certain SFPEs in an abstract setting. As an application of Theorem 2.9 we establish in Theorem 3.8 the existence of unique solutions of SFPEs associated with semilinear Kolmogorov PDEs with Lipschitz continuous nonlinearities even in the case where the associated semilinear Kolmogorov PDE does not possess a classical solution (see, for example, Hairer et al. [9]). To illustrate Theorem 3.8 in more detail we provide in the following result, Theorem 1.1 below, a special case of Theorem 3.8.

Theorem 1.1. *Let $d, m \in \mathbb{N}$, $L, T \in (0, \infty)$, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the standard scalar product on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be a norm on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be a norm on $\mathbb{R}^{d \times m}$, let $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ be locally Lipschitz continuous, let $f \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}^d, \mathbb{R})$ be at most polynomially growing, assume for all $t \in [0, T]$, $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $\max\{\langle x, \mu(x) \rangle, \|\sigma(x)\|^2\} \leq L(1 + \|x\|^2)$ and $|f(t, x, v) - f(t, x, w)| \leq L|v - w|$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, and for every $t \in [0, T]$, $x \in \mathbb{R}^d$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathbb{R}^d$ be an $(\mathbb{F}_s)_{s \in [t, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $s \in [t, T]$ it holds \mathbb{P} -a.s. that*

$$X_s^{t,x} = x + \int_t^s \mu(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r. \quad (1)$$

Then there exists a unique at most polynomially growing $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ such that for all $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that

$$u(t, x) = \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds \right]. \quad (2)$$

SFPEs of the form as in (2) have a strong connection with semilinear Kolmogorov PDEs and arise, for example, in models from the environmental sciences as well as in pricing problems from financial engineering (cf., for example, Burgard & Kjaer [2], Crépey et al. [3], Duffie et

al. [4], and Henry-Labordère [10]). SFPEs such as (2) are also important for full-history recursive multilevel Picard approximation (MLP) methods, which were recently introduced in [5, 11]; see also [1, 6, 12, 13]. In [11, 12] it has been shown that functions which satisfy SFPEs related to semilinear Kolmogorov PDEs can be approximated by MLP schemes without the curse of dimensionality. Theorem 1.1 above establishes existence of unique solutions of SFPEs related to semilinear Kolmogorov PDEs with Lipschitz continuous nonlinearities within the class of at most polynomially growing continuous functions. Theorem 1.1 is an immediate consequence of Corollary 3.10 in Section 3.4 below. Corollary 3.10, in turn, follows from Corollary 3.9 which itself is a special case of Theorem 3.8. Theorem 3.8 is an application of Theorem 2.9, the main result of this article. Theorem 3.8 shows the existence of unique solutions of SFPEs associated with suitable semilinear Kolmogorov PDEs with Lipschitz continuous nonlinearities within a certain class of continuous functions. Related existence and uniqueness results can be found, e.g., in Pazy [18, Theorem 6.1.2], Segal [20, Theorem 1], Weisler [22, Theorem 1], and Hutzenthaler et al. [11, Corollary 3.11].

The remainder of this article is organized as follows. In Section 2 we investigate SFPEs in an abstract setting. In Theorem 2.9 in Section 2.5, the main result of this article, we obtain under suitable assumptions an abstract existence and uniqueness result for solutions of SFPEs. Its proof is based on Banach's fixed point theorem. In Sections 2.1–2.3 we establish the well-definedness of the mapping to which Banach's fixed point theorem is applied in the proof of Theorem 2.9. In Section 2.4 we prove a Lipschitz estimate which establishes the contractivity property of the mapping to which Banach's fixed point theorem is applied in the proof of Theorem 2.9. In Section 3 we apply the abstract theory from Theorem 2.9 in Section 2 in the context of certain stochastic differential equations (SDEs) to obtain Theorem 3.8, the main result of Section 3. In Sections 3.1–3.3 we present several auxiliary results on certain SDEs in order to demonstrate that the hypotheses of Theorem 2.9 are satisfied in the setting of Theorem 3.8. The article is concluded by means of two simple corollaries of Theorem 3.8 (see Corollary 3.9 and Corollary 3.10 in Section 3.4 below).

2 Abstract stochastic fixed point equations (SFPEs)

In this section we study SFPEs from an abstract point of view. This section's main result is Theorem 2.9 below. It is an application of Banach's fixed point theorem to a suitable function. Corollary 2.7 in Section 2.3 establishes the well-definedness of this function. Corollary 2.7 is a direct consequence of Lemma 2.6 which we establish through an approximation argument building upon Lemmas 2.2 and 2.5. The contractivity property of the function to which we apply Banach's fixed point theorem in the proof Theorem 2.9 is established in Lemma 2.8 in Section 2.4 below.

2.1 Integrability properties for certain stochastic processes

Lemma 2.1. *Let $d \in \mathbb{N}$, $T \in (0, \infty)$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathcal{O}$ be $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable, let $g: \mathcal{O} \rightarrow \mathbb{R}$ be $\mathcal{B}(\mathcal{O})/\mathcal{B}(\mathbb{R})$ -measurable, let $h: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ be $\mathcal{B}([0, T] \times \mathcal{O})/\mathcal{B}(\mathbb{R})$ -measurable, let $V: [0, T] \times \mathcal{O} \rightarrow (0, \infty)$ be $\mathcal{B}([0, T] \times \mathcal{O})/\mathcal{B}((0, \infty))$ -measurable, and assume for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ that $\mathbb{E}[V(s, X_s^{t,x})] \leq V(t, x)$ and $\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left[\frac{|g(x)|}{V(T, x)} + \frac{|h(t, x)|}{V(t, x)} \right] < \infty$. Then it holds for all $t \in [0, T]$, $x \in \mathcal{O}$ that*

$$\mathbb{E} \left[|g(X_T^{t,x})| + \int_t^T |h(s, X_s^{t,x})| ds \right] < \infty. \quad (3)$$

Proof of Lemma 2.1. Throughout this proof let $c \in [0, \infty)$ satisfy for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$|g(x)| \leq cV(T, x) \quad \text{and} \quad |h(t, x)| \leq cV(t, x). \quad (4)$$

Observe that the hypothesis that $g: \mathcal{O} \rightarrow \mathbb{R}$ is $\mathcal{B}(\mathcal{O})/\mathcal{B}(\mathbb{R})$ -measurable, the hypothesis that $h: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ is $\mathcal{B}([0, T] \times \mathcal{O})/\mathcal{B}(\mathbb{R})$ -measurable, and the hypothesis that for every $t \in [0, T]$, $x \in \mathcal{O}$ it holds that $X^{t,x}: [t, T] \times \Omega \rightarrow \mathcal{O}$ is $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable ensure that for every $t \in [0, T]$, $x \in \mathcal{O}$ it holds that $\Omega \ni \omega \mapsto g(X_T^{t,x}(\omega)) \in \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable and $[t, T] \times \Omega \ni (s, \omega) \mapsto h(s, X_s^{t,x}(\omega)) \in \mathbb{R}$ is $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$ -measurable. The hypothesis that for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ it holds that $\mathbb{E}[V(s, X_s^{t,x})] \leq V(t, x)$, Fubini's theorem, and (4) hence ensure that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\begin{aligned} \mathbb{E} \left[|g(X_T^{t,x})| + \int_t^T |h(s, X_s^{t,x})| ds \right] &= \mathbb{E}[|g(X_T^{t,x})|] + \int_t^T \mathbb{E}[|h(s, X_s^{t,x})|] ds \\ &\leq \mathbb{E}[cV(T, X_T^{t,x})] + \int_t^T \mathbb{E}[cV(s, X_s^{t,x})] ds \leq cV(t, x) + \int_t^T cV(t, x) ds \\ &\leq c(1+T)V(t, x) < \infty. \end{aligned} \quad (5)$$

This demonstrates (3). The proof of Lemma 2.1 is thus completed. \square

2.2 Continuity properties for solutions of SFPEs

In this section we establish in Lemma 2.2, Lemma 2.3, and Corollary 2.4 several elementary convergence and approximation results. The convergence result in Lemma 2.2 and the approximation result in Corollary 2.4 pave the way for Section 2.3. They will together with Lemma 2.5 be employed in the proof of Lemma 2.6 in Section 2.3. Lemma 2.6, in turn, has Corollary 2.7 as a rather direct consequence, which itself is one of the cornerstones of the proof of Theorem 2.9.

Lemma 2.2. *Let $d \in \mathbb{N}$, $T \in (0, \infty)$, let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be a norm on \mathbb{R}^d , let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}: \|x\| \leq r\}$ and $\{y \in \mathbb{R}^d: \|y - x\| < 1/r\} \subseteq \mathcal{O}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathcal{O}$ be $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable, let $V \in C([0, T] \times \mathcal{O}, (0, \infty))$ satisfy for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ that $\mathbb{E}[V(s, X_s^{t,x})] \leq V(t, x)$, let $g_n \in C(\mathcal{O}, \mathbb{R})$, $n \in \mathbb{N}_0$, and $h_n \in C([0, T] \times \mathcal{O}, \mathbb{R})$, $n \in \mathbb{N}_0$, satisfy for all $n \in \mathbb{N}$ that $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} (\frac{|g_n(x)|}{V(T, x)} + \frac{|h_n(t, x)|}{V(t, x)})] = 0$, and assume that*

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|g_n(x) - g_0(x)|}{V(T, x)} + \frac{|h_n(t, x) - h_0(t, x)|}{V(t, x)} \right) \right] = 0. \quad (6)$$

Then

(i) *it holds for every $n \in \mathbb{N}_0$ that*

$$\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left[\frac{|g_n(x)|}{V(T, x)} + \frac{|h_n(t, x)|}{V(t, x)} \right] < \infty, \quad (7)$$

(ii) *it holds for every $n \in \mathbb{N}_0$ that there exists a unique $u_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ which satisfies for all $t \in [0, T]$, $x \in \mathcal{O}$ that*

$$u_n(t, x) = \mathbb{E} \left[g_n(X_T^{t,x}) + \int_t^T h_n(s, X_s^{t,x}) ds \right], \quad (8)$$

(iii) it holds that

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|u_n(t, x) - u_0(t, x)|}{V(t, x)} \right) \right] = 0, \quad (9)$$

and

(iv) it holds for every compact set $\mathcal{K} \subseteq \mathcal{O}$ that

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{K}} |u_n(t, x) - u_0(t, x)| \right] = 0. \quad (10)$$

Proof of Lemma 2.2. First, observe that for every $r \in (0, \infty)$ it holds that O_r is a compact set. This and the fact that for every $n \in \mathbb{N}_0$ it holds that $\mathcal{O} \ni x \mapsto \frac{g_n(x)}{V(T, x)} \in \mathbb{R}$ and $[0, T] \times \mathcal{O} \ni (t, x) \mapsto \frac{h_n(t, x)}{V(t, x)} \in \mathbb{R}$ are continuous imply that for all $n \in \mathbb{N}_0$, $r \in (0, \infty)$ it holds that

$$\sup \left(\left\{ \frac{|g_n(x)|}{V(T, x)} + \frac{|h_n(t, x)|}{V(t, x)} : t \in [0, T], x \in O_r \right\} \cup \{0\} \right) < \infty. \quad (11)$$

The hypothesis that for every $n \in \mathbb{N}$ it holds that $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} (\frac{|g_n(x)|}{V(T, x)} + \frac{|h_n(t, x)|}{V(t, x)})] = 0$ hence ensures that for every $n \in \mathbb{N}$ it holds that

$$\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left[\frac{|g_n(x)|}{V(T, x)} + \frac{|h_n(t, x)|}{V(t, x)} \right] < \infty. \quad (12)$$

Combining this with (6) demonstrates Item (i). Next observe that Item (i) and Lemma 2.1 establish Item (ii). Next note that the hypothesis that for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ it holds that $\mathbb{E}[V(s, X_s^{t, x})] \leq V(t, x)$ ensures that for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\begin{aligned} \frac{\mathbb{E}[|g_n(X_T^{t, x}) - g_0(X_T^{t, x})|]}{V(t, x)} &= \mathbb{E} \left[\frac{|g_n(X_T^{t, x}) - g_0(X_T^{t, x})|}{V(T, X_T^{t, x})} \cdot \frac{V(T, X_T^{t, x})}{V(t, x)} \right] \\ &\leq \left[\sup_{y \in \mathcal{O}} \left(\frac{|g_n(y) - g_0(y)|}{V(T, y)} \right) \right] \cdot \frac{\mathbb{E}[V(T, X_T^{t, x})]}{V(t, x)} \leq \sup_{y \in \mathcal{O}} \left(\frac{|g_n(y) - g_0(y)|}{V(T, y)} \right). \end{aligned} \quad (13)$$

This and (6) establish that

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|\mathbb{E}[g_n(X_T^{t, x})] - \mathbb{E}[g_0(X_T^{t, x})]|}{V(t, x)} \right) \right] = 0. \quad (14)$$

Furthermore, note that the hypothesis that for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ it holds that $\mathbb{E}[V(s, X_s^{t, x})] \leq V(t, x)$ assures that for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\begin{aligned} &\frac{\mathbb{E} \left[\int_t^T |h_n(s, X_s^{t, x}) - h_0(s, X_s^{t, x})| ds \right]}{V(t, x)} \\ &= \int_t^T \mathbb{E} \left[\frac{|h_n(s, X_s^{t, x}) - h_0(s, X_s^{t, x})|}{V(s, X_s^{t, x})} \cdot \frac{V(s, X_s^{t, x})}{V(t, x)} \right] ds \\ &\leq \int_t^T \left[\sup_{r \in [0, T]} \sup_{y \in \mathcal{O}} \left(\frac{|h_n(r, y) - h_0(r, y)|}{V(r, y)} \right) \right] \cdot \frac{\mathbb{E}[V(s, X_s^{t, x})]}{V(t, x)} ds \\ &\leq T \cdot \left[\sup_{r \in [0, T]} \sup_{y \in \mathcal{O}} \left(\frac{|h_n(r, y) - h_0(r, y)|}{V(r, y)} \right) \right]. \end{aligned} \quad (15)$$

This and (6) imply that

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{\left| \mathbb{E} \left[\int_t^T h_n(s, X_s^{t,x}) ds \right] - \mathbb{E} \left[\int_t^T h_0(s, X_s^{t,x}) ds \right] \right|}{V(t, x)} \right) \right] = 0. \quad (16)$$

The triangle inequality, (8), and (14) hence yield that

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|u_n(t, x) - u_0(t, x)|}{V(t, x)} \right) \right] = 0. \quad (17)$$

This establishes Item (iii). Moreover, observe that Item (iii) and the fact that $V: [0, T] \times \mathcal{O} \rightarrow (0, \infty)$ is continuous imply for every compact set $\mathcal{K} \subseteq \mathcal{O}$ that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{K}} |u_n(t, x) - u_0(t, x)| \right] \\ \leq \limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{K}} \left(\frac{|u_n(t, x) - u_0(t, x)|}{V(t, x)} \right) \right] \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{K}} V(t, x) \right] = 0. \end{aligned} \quad (18)$$

This establishes Item (iv). The proof of Lemma 2.2 is thus completed. \square

Lemma 2.3. *Let $d \in \mathbb{N}$, $T \in (0, \infty)$, let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be a norm on \mathbb{R}^d , let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}: \|x\| \leq r\}$ and $\{y \in \mathbb{R}^d: \|y - x\| < 1/r\} \subseteq \mathcal{O}$, and let $h \in C([0, T] \times \mathcal{O}, \mathbb{R})$ satisfy $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} |h(t, x)|] = 0$. Then there exist compactly supported $\mathfrak{h}_n \in C([0, T] \times \mathcal{O}, \mathbb{R})$, $n \in \mathbb{N}$, which satisfy that*

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |\mathfrak{h}_n(t, x) - h(t, x)| \right] = 0. \quad (19)$$

Proof of Lemma 2.3. Throughout this proof let $U_n \subseteq \mathcal{O}$, $n \in \mathbb{N}$, be the sets given by $U_n = \{x \in \mathcal{O}: (\exists z \in O_n: \|z - x\| < \frac{1}{2n})\}$. Note that for every $n \in \mathbb{N}$ it holds that $O_n \subseteq \mathcal{O}$ is a compact set, it holds that $U_n \subseteq \mathbb{R}^d$ is an open set which satisfies $U_n \subseteq \mathcal{O}$, and it holds that $O_n \subseteq U_n$. Urysohn's lemma (cf., for example, Rudin [19, Lemma 2.12]) hence ensures for every $n \in \mathbb{N}$ that there exists $\varphi_n \in C([0, T] \times \mathcal{O}, \mathbb{R})$ which satisfies for all $t \in [0, T]$, $x \in \mathcal{O}$ that $\mathbb{1}_{[0, T] \times O_n}(t, x) \leq \varphi_n(t, x) \leq \mathbb{1}_{[0, T] \times U_n}(t, x)$. Observe, in particular, that this implies that the functions $\varphi_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, have compact supports. In the next step we let $\mathfrak{h}_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ that $\mathfrak{h}_n(t, x) = \varphi_n(t, x)h(t, x)$. Note that this and the fact that for every $n \in \mathbb{N}$ it holds that $\varphi_n \in C([0, T] \times \mathcal{O}, \mathbb{R})$ is compactly supported imply that for every $n \in \mathbb{N}$ it holds that $\mathfrak{h}_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ is a compactly supported continuous function. Moreover, observe that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |\mathfrak{h}_n(t, x) - h(t, x)| \right] \\ = \limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} ([1 - \varphi_n(t, x)]|h(t, x)|) \right] \leq \limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_n} |h(t, x)| \right] = 0. \end{aligned} \quad (20)$$

This establishes (19). The proof of Lemma 2.3 is thus completed. \square

Corollary 2.4. *Let $d \in \mathbb{N}$, $T \in (0, \infty)$, let $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$ be a norm on \mathbb{R}^d , let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O} : \|x\| \leq r \text{ and } \{y \in \mathbb{R}^d : \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $h \in C([0, T] \times \mathcal{O}, \mathbb{R})$, $V \in C([0, T] \times \mathcal{O}, (0, \infty))$, and assume that $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} (\frac{|h(t, x)|}{V(t, x)})] = 0$. Then there exist compactly supported $\mathfrak{h}_n \in C([0, T] \times \mathcal{O}, \mathbb{R})$, $n \in \mathbb{N}$, which satisfy that*

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|\mathfrak{h}_n(t, x) - h(t, x)|}{V(t, x)} \right) \right] = 0. \quad (21)$$

Proof of Corollary 2.4. Throughout this proof let $g : [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ satisfy for all $t \in [0, T]$, $x \in \mathcal{O}$ that $g(t, x) = \frac{h(t, x)}{V(t, x)}$. Observe that the assumption that $h \in C([0, T] \times \mathcal{O}, \mathbb{R})$, the assumption that $V \in C([0, T] \times \mathcal{O}, (0, \infty))$, and the assumption that $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} (\frac{|h(t, x)|}{V(t, x)})] = 0$ prove that $g \in C([0, T] \times \mathcal{O}, \mathbb{R})$ and

$$\inf_{r \in (0, \infty)} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} |g(t, x)| \right] = 0. \quad (22)$$

Lemma 2.3 (with $h = g$ in the notation of Lemma 2.3) therefore ensures that there exist compactly supported $\mathfrak{g}_n \in C([0, T] \times \mathcal{O}, \mathbb{R})$, $n \in \mathbb{N}$, which satisfy that

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |\mathfrak{g}_n(t, x) - g(t, x)| \right] = 0. \quad (23)$$

Next let $\mathfrak{h}_n : [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ that $\mathfrak{h}_n(t, x) = \mathfrak{g}_n(t, x)V(t, x)$. Hence, we obtain that for all $n \in \mathbb{N}$ it holds that $\mathfrak{h}_n \in C([0, T] \times \mathcal{O}, \mathbb{R})$ and

$$\limsup_{k \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|\mathfrak{h}_k(t, x) - h(t, x)|}{V(t, x)} \right) \right] = \limsup_{k \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |\mathfrak{g}_k(t, x) - g(t, x)| \right] = 0. \quad (24)$$

This establishes (21). The proof of Corollary 2.4 is thus completed. \square

2.3 Regularity properties for solutions of SFPEs

In this section we establish Corollary 2.7, one of the building blocks of the proof of Theorem 2.9. Corollary 2.7 is a rather direct consequence of Lemma 2.6 which, in turn, we prove by means of an argument building upon Lemmas 2.2–2.5.

Lemma 2.5. *Let $d \in \mathbb{N}$, $T \in (0, \infty)$, let $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$ be a norm on \mathbb{R}^d , let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t, x} = (X_s^{t, x})_{s \in [t, T]} : [t, T] \times \Omega \rightarrow \mathcal{O}$ be $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable, let $g \in C(\mathcal{O}, \mathbb{R})$, $h \in C([0, T] \times \mathcal{O}, \mathbb{R})$ be bounded, and assume for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ and all $(\mathfrak{t}_n, \mathfrak{x}_n) \in [0, T] \times \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n \rightarrow \infty} [\|\mathfrak{t}_n - \mathfrak{t}_0\| + \|\mathfrak{x}_n - \mathfrak{x}_0\|] = 0$ that $\limsup_{n \rightarrow \infty} [\mathbb{P}(\|X_{\max\{s, \mathfrak{t}_n\}}^{\mathfrak{t}_n, \mathfrak{x}_n} - X_{\max\{s, \mathfrak{t}_0\}}^{\mathfrak{t}_0, \mathfrak{x}_0}\| \geq \varepsilon)] = 0$. Then*

(i) *it holds for all $t \in [0, T]$, $x \in \mathcal{O}$ that*

$$\mathbb{E} \left[|g(X_T^{t, x})| + \int_t^T |h(s, X_s^{t, x})| ds \right] < \infty \quad (25)$$

and

(ii) it holds that

$$[0, T] \times \mathcal{O} \ni (t, x) \mapsto \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T h(s, X_s^{t,x}) ds \right] \in \mathbb{R} \quad (26)$$

is continuous.

Proof of Lemma 2.5. Throughout this proof let $(t_n, \mathfrak{r}_n) \in [0, T] \times \mathcal{O}$, $n \in \mathbb{N}_0$, satisfy $\limsup_{n \rightarrow \infty} [|\mathfrak{t}_n - \mathfrak{t}_0| + \|\mathfrak{r}_n - \mathfrak{r}_0\|] = 0$. Note that Lemma 2.1 establishes Item (i). Next we prove Item (ii). For this we intend to show that

$$\limsup_{n \rightarrow \infty} \left[\mathbb{E}[|g(X_T^{t_n, \mathfrak{r}_n}) - g(X_T^{t_0, \mathfrak{r}_0})|] + \left| \mathbb{E} \left[\int_{t_n}^T h(s, X_s^{t_n, \mathfrak{r}_n}) ds \right] - \mathbb{E} \left[\int_{t_0}^T h(s, X_s^{t_0, \mathfrak{r}_0}) ds \right] \right| \right] = 0. \quad (27)$$

Next note that the fact that $g: \mathcal{O} \rightarrow \mathbb{R}$ and $h: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ are continuous ensures that for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ it holds that

$$\limsup_{n \rightarrow \infty} \left[\mathbb{P}(|g(X_T^{t_n, \mathfrak{r}_n}) - g(X_T^{t_0, \mathfrak{r}_0})| + |h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) - h(s, X_{\max\{s, t_0\}}^{t_0, \mathfrak{r}_0})| \geq \varepsilon) \right] = 0 \quad (28)$$

(cf., for example, Kallenberg [14, Lemma 4.3]). Combining this and the fact that $g: \mathcal{O} \rightarrow \mathbb{R}$ and $h: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ are bounded with Vitali's convergence theorem ensures that for all $s \in [0, T]$ it holds that

$$\limsup_{n \rightarrow \infty} \left(\mathbb{E}[|g(X_T^{t_n, \mathfrak{r}_n}) - g(X_T^{t_0, \mathfrak{r}_0})|] + \mathbb{E} \left[|h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) - h(s, X_{\max\{s, t_0\}}^{t_0, \mathfrak{r}_0})| \right] \right) = 0. \quad (29)$$

Lebesgue's dominated convergence theorem and the fact that $h: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ is bounded hence imply that

$$\limsup_{n \rightarrow \infty} \int_{t_0}^T \mathbb{E} \left[|h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) - h(s, X_{\max\{s, t_0\}}^{t_0, \mathfrak{r}_0})| \right] ds = 0. \quad (30)$$

This yields that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \mathbb{E} \left[\int_{t_n}^T h(s, X_s^{t_n, \mathfrak{r}_n}) ds \right] - \mathbb{E} \left[\int_{t_0}^T h(s, X_s^{t_0, \mathfrak{r}_0}) ds \right] \right| \\ &= \limsup_{n \rightarrow \infty} \left| \mathbb{E} \left[\int_{t_n}^T h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) ds \right] - \mathbb{E} \left[\int_{t_0}^T h(s, X_{\max\{s, t_0\}}^{t_0, \mathfrak{r}_0}) ds \right] \right| \\ &= \limsup_{n \rightarrow \infty} \left| \mathbb{E} \left[\int_{t_n}^{t_0} h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) ds + \int_{t_0}^T (h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) - h(s, X_{\max\{s, t_0\}}^{t_0, \mathfrak{r}_0})) ds \right] \right| \\ &\leq \limsup_{n \rightarrow \infty} \left(\mathbb{E} \left[\left| \int_{t_n}^{t_0} h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) ds \right| \right] + \mathbb{E} \left[\int_{t_0}^T |h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) - h(s, X_{\max\{s, t_0\}}^{t_0, \mathfrak{r}_0})| ds \right] \right) \\ &\leq \limsup_{n \rightarrow \infty} \left(|t_n - t_0| \left[\sup_{s \in [0, T]} \sup_{y \in \mathcal{O}} |h(s, y)| \right] + \int_{t_0}^T \mathbb{E} \left[|h(s, X_{\max\{s, t_n\}}^{t_n, \mathfrak{r}_n}) - h(s, X_{\max\{s, t_0\}}^{t_0, \mathfrak{r}_0})| \right] ds \right) \\ &= 0. \end{aligned} \quad (31)$$

Combining this with (29) demonstrates (27). The proof of Lemma 2.5 is thus completed. \square

Lemma 2.6. Let $d \in \mathbb{N}$, $T \in (0, \infty)$, let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be a norm on \mathbb{R}^d , let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}: \|x\| \leq r \text{ and } \{y \in \mathbb{R}^d: \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathcal{O}$ be $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable, assume for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ and all $(t_n, x_n) \in [0, T] \times \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n \rightarrow \infty} [|\mathfrak{t}_n - \mathfrak{t}_0| + \|\mathfrak{r}_n - \mathfrak{r}_0\|] = 0$

$t_0| + \|x_n - x_0\| = 0$ that $\limsup_{n \rightarrow \infty} [\mathbb{P}(\|X_{\max\{s, t_n\}}^{t_n, x_n} - X_{\max\{s, t_0\}}^{t_0, x_0}\| \geq \varepsilon)] = 0$, let $g \in C(\mathcal{O}, \mathbb{R})$, $h \in C([0, T] \times \mathcal{O}, \mathbb{R})$, $V \in C([0, T] \times \mathcal{O}, (0, \infty))$ and $u: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ satisfy for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ that $\mathbb{E}[V(s, X_s^{t, x})] \leq V(t, x)$, and assume for all $t \in [0, T]$, $x \in \mathcal{O}$ that $\inf_{r \in (0, \infty)} [\sup_{s \in [0, T]} \sup_{y \in \mathcal{O} \setminus \mathcal{O}_r} (\frac{|g(y)|}{V(T, y)} + \frac{|h(s, y)|}{V(s, y)})] = 0$ and

$$u(t, x) = \mathbb{E} \left[g(X_T^{t, x}) + \int_t^T h(s, X_s^{t, x}) ds \right] \quad (32)$$

(cf. Item (ii) of Lemma 2.2). Then

(i) it holds that $u \in C([0, T] \times \mathcal{O}, \mathbb{R})$ and

(ii) it holds in the case of $\sup_{r \in (0, \infty)} [\inf_{t \in [0, T]} \inf_{x \in \mathcal{O} \setminus \mathcal{O}_r} V(t, x)] = \infty$ that

$$\lim_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} \left(\frac{|u(t, x)|}{V(t, x)} \right) \right] = 0. \quad (33)$$

Proof of Lemma 2.6. Throughout this proof let $\mathbf{g}_n: \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, and $\mathbf{h}_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be compactly supported continuous functions which satisfy that

$$\limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|\mathbf{g}_n(x) - g(x)|}{V(T, x)} + \frac{|\mathbf{h}_n(t, x) - h(t, x)|}{V(t, x)} \right) \right] = 0 \quad (34)$$

(cf. Corollary 2.4) and let $\mathbf{u}_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\mathbf{u}_n(t, x) = \mathbb{E} \left[\mathbf{g}_n(X_T^{t, x}) + \int_t^T \mathbf{h}_n(s, X_s^{t, x}) ds \right] \quad (35)$$

(cf. Lemma 2.1). Note that Lemma 2.5 assures for every $n \in \mathbb{N}$ that $\mathbf{u}_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ is continuous. Next observe that the fact that $\mathbf{g}_n: \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, and $\mathbf{h}_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, are compactly supported ensures that for every $n \in \mathbb{N}$ there exists $r \in (0, \infty)$ which satisfies that for all $t \in [0, T]$, $x \in \mathcal{O} \setminus \mathcal{O}_r$ it holds that $\mathbf{g}_n(x) = 0 = \mathbf{h}_n(t, x)$. This implies for every $n \in \mathbb{N}$ that

$$\inf_{r \in (0, \infty)} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} \left(\frac{|\mathbf{g}_n(x)|}{V(T, x)} + \frac{|\mathbf{h}_n(t, x)|}{V(t, x)} \right) \right] = 0. \quad (36)$$

Item (iv) of Lemma 2.2, (34), and the fact that $\mathbf{u}_n: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, are continuous therefore imply that $u: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ is continuous. This establishes Item (i). In the next step we prove Item (ii). For this we assume that $\sup_{r \in (0, \infty)} [\inf_{t \in [0, T]} \inf_{x \in \mathcal{O} \setminus \mathcal{O}_r} V(t, x)] = \infty$. Note that this entails for every $n \in \mathbb{N}$ that

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} \left(\frac{|\mathbf{u}_n(t, x)|}{V(t, x)} \right) \right] \\ & \leq \limsup_{r \rightarrow \infty} \left(\left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |\mathbf{u}_n(t, x)| \right] \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} \left(\frac{1}{V(t, x)} \right) \right] \right) \\ & \leq \limsup_{r \rightarrow \infty} \left(\left[\left(\sup_{x \in \mathcal{O}} |\mathbf{g}_n(x)| \right) + T \left(\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |\mathbf{h}_n(t, x)| \right) \right] \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} \left(\frac{1}{V(t, x)} \right) \right] \right) = 0. \end{aligned} \quad (37)$$

Combining this with Item (iii) of Lemma 2.2 yields that

$$\begin{aligned}
& \limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} \left(\frac{|u(t, x)|}{V(t, x)} \right) \right] \\
& \leq \inf_{n \in \mathbb{N}} \left(\limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} \left(\frac{|u(t, x) - \mathbf{u}_n(t, x)| + |\mathbf{u}_n(t, x)|}{V(t, x)} \right) \right] \right) \\
& = \inf_{n \in \mathbb{N}} \left(\limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} \left(\frac{|u(t, x) - \mathbf{u}_n(t, x)|}{V(t, x)} \right) \right] \right) \\
& \leq \inf_{n \in \mathbb{N}} \left(\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|u(t, x) - \mathbf{u}_n(t, x)|}{V(t, x)} \right) \right) \\
& \leq \limsup_{n \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{|u(t, x) - \mathbf{u}_n(t, x)|}{V(t, x)} \right) \right] = 0.
\end{aligned} \tag{38}$$

This establishes Item (ii). The proof of Lemma 2.6 is thus completed. \square

Lemma 2.6 allows to infer the next result, Corollary 2.7, which constitutes an important ingredient of the proof of Theorem 2.9.

Corollary 2.7. *Let $d \in \mathbb{N}$, $L, T \in (0, \infty)$, let $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$ be a norm on \mathbb{R}^d , let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O} : \|x\| \leq r\}$ and $\{y \in \mathbb{R}^d : \|y - x\| < 1/r\} \subseteq \mathcal{O}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]} : [t, T] \times \Omega \rightarrow \mathcal{O}$ be $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable, assume for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ and all $(t_n, x_n) \in [0, T] \times \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n \rightarrow \infty} [|t_n - t_0| + \|x_n - x_0\|] = 0$ that $\limsup_{n \rightarrow \infty} [\mathbb{P}(\|X_{\max\{s, t_n\}}^{t_n, x_n} - X_{\max\{s, t_0\}}^{t_0, x_0}\| \geq \varepsilon)] = 0$, let $f \in C([0, T] \times \mathcal{O} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathcal{O}, \mathbb{R})$, $u \in C([0, T] \times \mathcal{O}, \mathbb{R})$, $V \in C([0, T] \times \mathcal{O}, (0, \infty))$ satisfy for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ that $\mathbb{E}[V(s, X_s^{t,x})] \leq V(t, x)$, and assume for all $t \in [0, T]$, $x \in \mathcal{O}$, $v, w \in \mathbb{R}$ that $\inf_{r \in (0, \infty)} [\sup_{s \in [0, T]} \sup_{y \in \mathcal{O} \setminus O_r} (\frac{|f(s, y, 0)| + |u(s, y)|}{V(s, y)} + \frac{|g(y)|}{V(T, y)})] = 0$ and $|f(t, x, v) - f(t, x, w)| \leq L|v - w|$. Then*

(i) *it holds for all $t \in [0, T]$, $x \in \mathcal{O}$ that*

$$\mathbb{E} \left[|g(X_T^{t,x})| + \int_t^T |f(s, X_s^{t,x}, u(s, X_s^{t,x}))| ds \right] < \infty, \tag{39}$$

(ii) *it holds that*

$$[0, T] \times \mathcal{O} \ni (t, x) \mapsto \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds \right] \in \mathbb{R} \tag{40}$$

is continuous, and

(iii) *it holds in the case of $\sup_{r \in (0, \infty)} [\inf_{t \in [0, T]} \inf_{x \in \mathcal{O} \setminus O_r} V(t, x)] = \infty$ that*

$$\lim_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} \left(\frac{\left| \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds \right] \right|}{V(t, x)} \right) \right] = 0. \tag{41}$$

Proof of Corollary 2.7. First, observe that

$$[0, T] \times \mathcal{O} \ni (t, x) \mapsto f(t, x, u(t, x)) \in \mathbb{R} \quad (42)$$

is a continuous function which satisfies for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$|f(t, x, u(t, x))| \leq |f(t, x, 0)| + L|u(t, x)|. \quad (43)$$

The hypothesis that $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} (\frac{|f(t, x, 0)| + |u(t, x)|}{V(t, x)})] = 0$ therefore ensures that

$$\begin{aligned} & \inf_{r \in (0, \infty)} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} \left(\frac{|f(t, x, u(t, x))|}{V(t, x)} \right) \right] \\ & \leq \inf_{r \in (0, \infty)} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} \left(\frac{|f(t, x, 0)|}{V(t, x)} + L \frac{|u(t, x)|}{V(t, x)} \right) \right] = 0. \end{aligned} \quad (44)$$

Lemma 2.1 and Item (i) of Lemma 2.2 hence establish Item (i). Moreover, Lemma 2.6 (with $g = g$, $h = ([0, T] \times \mathcal{O} \ni (t, x) \mapsto f(t, x, u(t, x)) \in \mathbb{R})$, $u = ([0, T] \times \mathcal{O} \ni (t, x) \mapsto \mathbb{E}[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds] \in \mathbb{R})$ in the notation of Lemma 2.6) establishes Items (ii) and (iii). The proof of Corollary 2.7 is thus completed. \square

2.4 Contractivity properties for SFPEs

In this section we establish an elementary Lipschitz estimate (see Lemma 2.8 below) which will yield the contractivity needed in the proof of Theorem 2.9.

Lemma 2.8. *Let $d \in \mathbb{N}$, $L, T \in (0, \infty)$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathcal{O}$ be $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable, let $V: [0, T] \times \mathcal{O} \rightarrow (0, \infty)$ be $\mathcal{B}([0, T] \times \mathcal{O})/\mathcal{B}((0, \infty))$ -measurable, assume for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ that $\mathbb{E}[V(s, X_s^{t,x})] \leq V(t, x)$, let $f: [0, T] \times \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$ be $\mathcal{B}([0, T] \times \mathcal{O} \times \mathbb{R})/\mathcal{B}(\mathbb{R})$ -measurable, assume for all $t \in [0, T]$, $x \in \mathcal{O}$, $v, w \in \mathbb{R}$ that $|f(t, x, v) - f(t, x, w)| \leq L|v - w|$, let $v, w: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ be $\mathcal{B}([0, T] \times \mathcal{O})/\mathcal{B}(\mathbb{R})$ -measurable, and assume that*

$$\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left[\frac{|v(t, x)| + |w(t, x)|}{V(t, x)} \right] < \infty. \quad (45)$$

Then it holds for all $\lambda \in (0, \infty)$, $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\begin{aligned} & \mathbb{E} \left[\int_t^T |f(s, X_s^{t,x}, v(s, X_s^{t,x})) - f(s, X_s^{t,x}, w(s, X_s^{t,x}))| ds \right] \\ & \leq \frac{L}{\lambda} e^{-\lambda t} V(t, x) \left[\sup_{s \in [0, T]} \sup_{y \in \mathcal{O}} \left(\frac{e^{\lambda s} |v(s, y) - w(s, y)|}{V(s, y)} \right) \right]. \end{aligned} \quad (46)$$

Proof of Lemma 2.8. First, note that the fact that $f: [0, T] \times \mathcal{O} \times \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{B}([0, T] \times \mathcal{O} \times \mathbb{R})/\mathcal{B}(\mathbb{R})$ -measurable, the fact that $v, w: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$ are $\mathcal{B}([0, T] \times \mathcal{O})/\mathcal{B}(\mathbb{R})$ -measurable, and the fact that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that $X^{t,x}: [t, T] \times \Omega \rightarrow \mathcal{O}$ is $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable ensure that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$[t, T] \times \Omega \ni (s, \omega) \mapsto |f(s, X_s^{t,x}(\omega), v(s, X_s^{t,x}(\omega))) - f(s, X_s^{t,x}(\omega), w(s, X_s^{t,x}(\omega)))| \in \mathbb{R} \quad (47)$$

is $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$ -measurable. Next observe that the hypothesis that for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ it holds that $\mathbb{E}[V(s, X_s^{t,x})] \leq V(t, x)$, the hypothesis that for all $t \in [0, T]$, $x \in \mathcal{O}$,

$a, b \in \mathbb{R}$ it holds that $|f(t, x, a) - f(t, x, b)| \leq L|a - b|$, and Fubini's theorem ensure that for all $\lambda \in (0, \infty)$, $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\begin{aligned}
& \mathbb{E} \left[\int_t^T |f(s, X_s^{t,x}, v(s, X_s^{t,x})) - f(s, X_s^{t,x}, w(s, X_s^{t,x}))| ds \right] \\
& \leq \mathbb{E} \left[\int_t^T L |v(s, X_s^{t,x}) - w(s, X_s^{t,x})| ds \right] \\
& = L \int_t^T \mathbb{E} \left[\frac{e^{\lambda s} |v(s, X_s^{t,x}) - w(s, X_s^{t,x})|}{V(s, X_s^{t,x})} V(s, X_s^{t,x}) \right] e^{-\lambda s} ds \\
& \leq L \int_t^T \left[\sup_{r \in [0, T]} \sup_{y \in \mathcal{O}} \left(\frac{e^{\lambda r} |v(r, y) - w(r, y)|}{V(r, y)} \right) \right] \mathbb{E}[V(s, X_s^{t,x})] e^{-\lambda s} ds \\
& \leq L \left[\sup_{r \in [0, T]} \sup_{y \in \mathcal{O}} \left(\frac{e^{\lambda r} |v(r, y) - w(r, y)|}{V(r, y)} \right) \right] V(t, x) \int_t^T e^{-\lambda s} ds \\
& \leq \frac{L}{\lambda} \left[\sup_{r \in [0, T]} \sup_{y \in \mathcal{O}} \left(\frac{e^{\lambda r} |v(r, y) - w(r, y)|}{V(r, y)} \right) \right] e^{-\lambda t} V(t, x).
\end{aligned} \tag{48}$$

This establishes (46). The proof of Lemma 2.8 is thus completed. \square

2.5 Existence and uniqueness properties for solutions of SFPEs

Combining Banach's fixed point theorem with Corollary 2.7 and Lemma 2.8 allows to conclude the main result of this section, Theorem 2.9 below.

Theorem 2.9. *Let $d \in \mathbb{N}$, $L, T \in (0, \infty)$, let $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$ be a norm on \mathbb{R}^d , let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O} : \|x\| \leq r \text{ and } \{y \in \mathbb{R}^d : \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]} : [t, T] \times \Omega \rightarrow \mathcal{O}$ be $(\mathcal{B}([t, T]) \otimes \mathcal{F})/\mathcal{B}(\mathcal{O})$ -measurable, assume for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ and all $(t_n, x_n) \in [0, T] \times \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n \rightarrow \infty} [|t_n - t_0| + \|x_n - x_0\|] = 0$ that $\limsup_{n \rightarrow \infty} [\mathbb{P}(\|X_{\max\{s, t_n\}}^{t_n, x_n} - X_{\max\{s, t_0\}}^{t_0, x_0}\| \geq \varepsilon)] = 0$, let $f \in C([0, T] \times \mathcal{O} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathcal{O}, \mathbb{R})$, $V \in C([0, T] \times \mathcal{O}, (0, \infty))$ satisfy for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$, $v, w \in \mathbb{R}$ that $\mathbb{E}[V(s, X_s^{t,x})] \leq V(t, x)$ and $|f(t, x, v) - f(t, x, w)| \leq L|v - w|$, and assume that $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} (\frac{|f(t, x, 0)|}{V(t, x)} + \frac{|g(x)|}{V(T, x)})] = 0$ and $\sup_{r \in (0, \infty)} [\inf_{t \in [0, T]} \inf_{x \in \mathcal{O} \setminus O_r} V(t, x)] = \infty$. Then there exists a unique $u \in C([0, T] \times \mathcal{O}, \mathbb{R})$ such that*

(i) it holds that

$$\limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} \left(\frac{|u(t, x)|}{V(t, x)} \right) \right] = 0 \tag{49}$$

and

(ii) it holds for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$u(t, x) = \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds \right]. \tag{50}$$

Proof of Theorem 2.9. Throughout this proof let \mathcal{V} be the set given by

$$\mathcal{V} = \left\{ u \in C([0, T] \times \mathcal{O}, \mathbb{R}) : \limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} \left(\frac{|u(t, x)|}{V(t, x)} \right) \right] = 0 \right\}, \tag{51}$$

let \mathcal{W}_1 and \mathcal{W}_2 be the sets given by

$$\mathcal{W}_1 = \left\{ u \in C([0, T] \times \mathcal{O}, \mathbb{R}) : \sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |u(t, x)| < \infty \right\} \quad (52)$$

and

$$\mathcal{W}_2 = \left\{ u \in C([0, T] \times \mathcal{O}, \mathbb{R}) : \limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} |u(t, x)| \right] = 0 \right\}, \quad (53)$$

let $\|\cdot\|_\lambda : \mathcal{V} \rightarrow [0, \infty)$, $\lambda \in \mathbb{R}$, satisfy for every $\lambda \in \mathbb{R}$, $v \in \mathcal{V}$ that

$$\|v\|_\lambda = \sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \left(\frac{e^{\lambda t} |v(t, x)|}{V(t, x)} \right) \quad (54)$$

(see Item (i) of Lemma 2.2), and let $\|\cdot\|_{\mathcal{W}_i} : \mathcal{W}_i \rightarrow [0, \infty)$, $i \in \{1, 2\}$, satisfy for every $i \in \{1, 2\}$, $w \in \mathcal{W}_i$ that

$$\|w\|_{\mathcal{W}_i} = \sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |w(t, x)|. \quad (55)$$

Recall that $(\mathcal{W}_1, \|\cdot\|_{\mathcal{W}_1})$ is an \mathbb{R} -Banach space. Combining this with the fact that \mathcal{W}_2 is a closed subset of $(\mathcal{W}_1, \|\cdot\|_{\mathcal{W}_1})$ (see Lemma 2.2) implies that $(\mathcal{W}_2, \|\cdot\|_{\mathcal{W}_2})$ is an \mathbb{R} -Banach space. Moreover, observe that $(\mathcal{V}, \|\cdot\|_\lambda)$, $\lambda \in \mathbb{R}$, are normed \mathbb{R} -vector spaces. In the next step we show that $(\mathcal{V}, \|\cdot\|_0)$ is complete. For this let $v_n \in \mathcal{V}$, $n \in \mathbb{N}$, satisfy $\limsup_{n \rightarrow \infty} [\sup_{m \geq n} \|v_n - v_m\|_0] = 0$. This implies that $\frac{v_n}{V} : [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, is a Cauchy sequence in $(\mathcal{W}_2, \|\cdot\|_{\mathcal{W}_2})$. Thus, there exists $\phi \in \mathcal{W}_2$ which satisfies that $\limsup_{n \rightarrow \infty} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} |\frac{v_n(t, x)}{V(t, x)} - \phi(t, x)|] = 0$. Hence, we obtain that $\phi V = ([0, T] \times \mathcal{O} \ni (t, x) \mapsto \phi(t, x)V(t, x) \in \mathbb{R}) \in \mathcal{V}$ and $\limsup_{n \rightarrow \infty} \|v_n - \phi V\|_0 = 0$. This demonstrates that $(\mathcal{V}, \|\cdot\|_0)$ is an \mathbb{R} -Banach space. Combining this with the fact that for every $\nu \in \mathbb{R}$, $\lambda \in [\nu, \infty)$, $v \in \mathcal{V}$ it holds that $\|v\|_\nu \leq \|v\|_\lambda \leq e^{(\lambda - \nu)T} \|v\|_\nu$ shows that for every $\lambda \in \mathbb{R}$ it holds that $(\mathcal{V}, \|\cdot\|_\lambda)$ is an \mathbb{R} -Banach space. Next note that Corollary 2.7 yields that there exists a unique $\Phi : \mathcal{V} \rightarrow \mathcal{V}$ which satisfies for all $t \in [0, T]$, $x \in \mathcal{O}$, $v \in \mathcal{V}$ that

$$[\Phi(v)](t, x) = \mathbb{E} \left[g(X_T^{t, x}) + \int_t^T f(s, X_s^{t, x}, v(s, X_s^{t, x})) ds \right]. \quad (56)$$

Moreover, observe that Lemma 2.8 ensures for all $\lambda \in (0, \infty)$, $v, w \in \mathcal{V}$ that

$$\|\Phi(v) - \Phi(w)\|_\lambda \leq \frac{L}{\lambda} \|v - w\|_\lambda. \quad (57)$$

Hence, we obtain for all $\lambda \in [2L, \infty)$, $v, w \in \mathcal{V}$ that

$$\|\Phi(v) - \Phi(w)\|_\lambda \leq \frac{1}{2} \|v - w\|_\lambda. \quad (58)$$

Banach's fixed point theorem therefore demonstrates that there exists a unique $u \in \mathcal{V}$ which satisfies $\Phi(u) = u$. The proof of Theorem 2.9 is thus completed. \square

3 SFPEs associated with stochastic differential equations (SDEs)

In this section we apply the abstract existence and uniqueness result which we obtained in the previous section (see Theorem 2.9 in Section 2 above) to certain SDEs (see Section 3.4 below). In Sections 3.1–3.3 we present, for the reader's convenience and for the sake of completeness, some elementary and essentially well-known results on SDEs. These results are employed to show that the hypotheses of Theorem 2.9 are indeed satisfied in the setting of Theorem 3.8 (cf. Lemmas 3.1 and 3.7).

3.1 A priori estimates for solutions of SDEs

The following well-known result, Lemma 3.1 below (cf., for example, Gyöngy & Krylov [8]), can be seen as an extension of moment bounds for solutions of SDEs in the presence of a Lyapunov function or, in other words, a non-negative supersolution of the corresponding Kolmogorov PDE.

Lemma 3.1. *Let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be an open set, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the standard scalar product on \mathbb{R}^d , let $\mu \in C([0, T] \times \mathcal{O}, \mathbb{R}^d)$, $\sigma \in C([0, T] \times \mathcal{O}, \mathbb{R}^{d \times m})$, $V \in C^{1,2}([0, T] \times \mathcal{O}, [0, \infty))$ satisfy for all $t \in [0, T]$, $x \in \mathcal{O}$ that*

$$\left(\frac{\partial V}{\partial t}\right)(t, x) + \frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^* (\text{Hess}_x V)(t, x)) + \langle \mu(t, x), (\nabla_x V)(t, x) \rangle \leq 0, \quad (59)$$

let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, let $\tau: \Omega \rightarrow [0, T]$ be an $(\mathbb{F}_t)_{t \in [0, T]}$ -stopping time, and let $X: [0, T] \times \Omega \rightarrow \mathcal{O}$ be an $(\mathbb{F}_t)_{t \in [0, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s. \quad (60)$$

Then it holds that

$$\mathbb{E}[V(\tau, X_\tau)] \leq \mathbb{E}[V(0, X_0)]. \quad (61)$$

Proof of Lemma 3.1. Throughout this proof let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d \times m}$, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}: \|x\| \leq r \text{ and } \{y \in \mathbb{R}^d: \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $Y: [0, T] \times \Omega \rightarrow \mathbb{R}$ be an $(\mathbb{F}_t)_{t \in [0, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$Y_t = \int_0^t \langle (\nabla_x V)(s, X_s), \sigma(s, X_s) dW_s \rangle, \quad (62)$$

and let $\rho_n: \Omega \rightarrow [0, T]$, $n \in \mathbb{N}$, be the $(\mathbb{F}_t)_{t \in [0, T]}$ -stopping times which satisfy for all $n \in \mathbb{N}$ that

$$\rho_n = \inf(\{t \in [0, T]: X_t \notin O_n\} \cup \{T\}). \quad (63)$$

Observe that the fact that X has continuous sample paths and the fact that $[0, T]$ is compact ensure that for all $\omega \in \Omega$ it holds that $\{X_t(\omega): t \in [0, T]\}$ is compact. Combining this with the fact that $\mathbb{R}^d \ni x \mapsto \|x\| \in [0, \infty)$ and $\mathbb{R}^d \ni x \mapsto \inf(\{1\} \cup \{\|x - y\|: y \in \mathbb{R}^d \setminus \mathcal{O}\}) \in [0, 1]$ are continuous implies that for every $\omega \in \Omega$ there exist $\varepsilon, r \in (0, \infty)$ such that for all $t \in [0, T]$ it holds that $\{y \in \mathbb{R}^d: \|y - X_t(\omega)\| < \varepsilon\} \subseteq \mathcal{O}$ and $\sup_{t \in [0, T]} \|X_t(\omega)\| \leq r$. Combining this with the fact that for all $\varepsilon, r \in (0, \infty)$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq n$ it holds that $r \leq k$ and $1/k \leq \varepsilon$ implies that for every $\omega \in \Omega$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq n$ it holds that $\rho_k(\omega) = T$. Next note that the assumption that $\nabla_x V: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^d$ and $\sigma: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^{d \times m}$ are continuous implies that for all $n \in \mathbb{N}$ it holds that

$$\sup \left(\{ \|(\nabla_x V)(t, x)\| + \|\sigma(t, x)\| : t \in [0, T], x \in O_n \} \cup \{0\} \right) < \infty. \quad (64)$$

This yields for all $n \in \mathbb{N}$ that

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \|(\nabla_x V)(s, X_s)\|^2 \|\sigma(s, X_s)\|_{\{0 < s \leq \min\{\tau, \rho_n\}\}}^2 ds \right] \\ & \leq T \left[\sup \left(\{ \|(\nabla_x V)(t, x)\| + \|\sigma(t, x)\| : t \in [0, T], x \in O_n \} \cup \{0\} \right) \right]^4 < \infty. \end{aligned} \quad (65)$$

Combining (62) and (63) hence assures for all $n \in \mathbb{N}$ that

$$\mathbb{E}[Y_{\min\{\tau, \rho_n\}}] = \mathbb{E}\left[\int_0^T \langle (\nabla_x V)(s, X_s), \sigma(s, X_s) \mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} dW_s \rangle\right] = 0. \quad (66)$$

Next note that Itô's formula ensures that for all $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$\begin{aligned} & V(t, X_t) \\ &= V(0, X_0) + \int_0^t \left(\frac{\partial V}{\partial t}\right)(s, X_s) ds + \int_0^t \langle (\nabla_x V)(s, X_s), \sigma(s, X_s) dW_s \rangle \\ & \quad + \int_0^t \left[\langle (\nabla_x V)(s, X_s), \mu(s, X_s) \rangle + \frac{1}{2} \text{Trace}(\sigma(s, X_s)[\sigma(s, X_s)]^* (\text{Hess}_x V)(s, X_s)) \right] ds \\ &= V(0, X_0) + \int_0^t \left(\frac{\partial V}{\partial t}\right)(s, X_s) ds + Y_t \\ & \quad + \int_0^t \left[\langle \mu(s, X_s), (\nabla_x V)(s, X_s) \rangle + \frac{1}{2} \text{Trace}(\sigma(s, X_s)[\sigma(s, X_s)]^* (\text{Hess}_x V)(s, X_s)) \right] ds. \end{aligned} \quad (67)$$

This and the fact that X has continuous sample paths imply that for all $n \in \mathbb{N}$ it holds \mathbb{P} -a.s. that

$$\begin{aligned} & V(\min\{\tau, \rho_n\}, X_{\min\{\tau, \rho_n\}}) \\ &= V(0, X_0) + Y_{\min\{\tau, \rho_n\}} + \int_0^{\min\{\tau, \rho_n\}} \left(\frac{\partial V}{\partial t}\right)(s, X_s) ds + \int_0^{\min\{\tau, \rho_n\}} \langle \mu(s, X_s), (\nabla_x V)(s, X_s) \rangle ds \\ & \quad + \int_0^{\min\{\tau, \rho_n\}} \frac{1}{2} \text{Trace}(\sigma(s, X_s)[\sigma(s, X_s)]^* (\text{Hess}_x V)(s, X_s)) ds. \end{aligned} \quad (68)$$

This and (59) guarantee that for all $n \in \mathbb{N}$ it holds \mathbb{P} -a.s. that

$$V(\min\{\tau, \rho_n\}, X_{\min\{\tau, \rho_n\}}) \leq V(0, X_0) + Y_{\min\{\tau, \rho_n\}}. \quad (69)$$

Combining this and (66) yields for all $n \in \mathbb{N}$ that

$$\mathbb{E}[V(\min\{\tau, \rho_n\}, X_{\min\{\tau, \rho_n\}})] \leq \mathbb{E}[V(0, X_0)]. \quad (70)$$

Fatou's lemma hence ensures that

$$\begin{aligned} \mathbb{E}[V(\tau, X_\tau)] &= \mathbb{E}\left[\liminf_{n \rightarrow \infty} V(\min\{\tau, \rho_n\}, X_{\min\{\tau, \rho_n\}})\right] \\ &\leq \liminf_{n \rightarrow \infty} \mathbb{E}[V(\min\{\tau, \rho_n\}, X_{\min\{\tau, \rho_n\}})] \leq \mathbb{E}[V(0, X_0)]. \end{aligned} \quad (71)$$

The proof of Lemma 3.1 is thus completed. \square

The next elementary result, Lemma 3.2 below, provides a way to construct from a supersolution of a suitable elliptic PDE a supersolution of a Kolmogorov PDE (cf. Lemma 3.1 above). Later we will employ Lemma 3.2 to infer Corollary 3.9 from Theorem 3.8.

Lemma 3.2. *Let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, $\rho \in \mathbb{R}$, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the standard scalar product on \mathbb{R}^d , let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, let $\mu \in C([0, T] \times \mathcal{O}, \mathbb{R}^d)$, $\sigma \in C([0, T] \times \mathcal{O}, \mathbb{R}^{d \times m})$, $V \in C^2(\mathcal{O}, (0, \infty))$ satisfy for all $t \in [0, T]$, $x \in \mathcal{O}$ that*

$$\frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^* (\text{Hess} V)(x)) + \langle \mu(t, x), (\nabla V)(x) \rangle \leq \rho V(x), \quad (72)$$

and let $\mathbb{V}: [0, T] \times \mathcal{O} \rightarrow (0, \infty)$ satisfy for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\mathbb{V}(t, x) = e^{-\rho t} V(x). \quad (73)$$

Then

(i) it holds that $\mathbb{V} \in C^2([0, T] \times \mathcal{O}, (0, \infty))$ and

(ii) it holds for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\left(\frac{\partial \mathbb{V}}{\partial t}\right)(t, x) + \frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^*(\text{Hess}_x \mathbb{V})(t, x)) + \langle \mu(t, x), (\nabla_x \mathbb{V})(t, x) \rangle \leq 0. \quad (74)$$

Proof of Lemma 3.2. First, note that the chain rule and the fact that $V \in C^2(\mathcal{O}, (0, \infty))$ ensure for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$(I) \quad \mathbb{V} \in C^2([0, T] \times \mathcal{O}, \mathbb{R}),$$

$$(II) \quad \left(\frac{\partial \mathbb{V}}{\partial t}\right)(t, x) = -\rho e^{-\rho t} V(x) = -\rho \mathbb{V}(t, x),$$

$$(III) \quad (\nabla_x \mathbb{V})(t, x) = e^{-\rho t} (\nabla V)(x), \text{ and}$$

$$(IV) \quad (\text{Hess}_x \mathbb{V})(t, x) = e^{-\rho t} (\text{Hess } V)(x).$$

Note that Item (I) establishes Item (i). Moreover, combining (72) with Items (II)–(IV) yields for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\begin{aligned} & \left(\frac{\partial \mathbb{V}}{\partial t}\right)(t, x) + \frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^*(\text{Hess}_x \mathbb{V})(t, x)) + \langle \mu(t, x), (\nabla_x \mathbb{V})(t, x) \rangle \\ & = e^{-\rho t} \left(-\rho V(x) + \frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^*(\text{Hess } V)(x)) + \langle \mu(t, x), (\nabla V)(x) \rangle \right) \leq 0. \end{aligned} \quad (75)$$

This establishes Item (ii). The proof of Lemma 3.2 is thus completed. \square

The next elementary result, Lemma 3.3 below, establishes in conjunction with Lemma 3.2 above that under certain coercivity and linear growth conditions (see (76) in Lemma 3.3) Lyapunov-type functions with polynomial growth are available (cf. also Grohs et al. [7, Lemma 2.21]). Lemma 3.3 will later on allow to infer Corollary 3.10 from Corollary 3.9.

Lemma 3.3. *Let $d, m \in \mathbb{N}$, $c, T, p, \rho \in (0, \infty)$ satisfy $\rho = \frac{pc}{2} \max\{p + 1, 3\}$, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the standard scalar product on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d \times m}$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, and let $\mu: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^d$, $\sigma: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^{d \times m}$, $V: \mathcal{O} \rightarrow (0, \infty)$ satisfy for all $t \in [0, T]$, $x \in \mathcal{O}$ that*

$$\max\{\langle x, \mu(t, x) \rangle, \|\sigma(t, x)\|^2\} \leq c(1 + \|x\|^2) \quad \text{and} \quad V(x) = (1 + \|x\|^2)^{p/2}. \quad (76)$$

Then

(i) it holds that $V \in C^\infty(\mathcal{O}, (0, \infty))$ and

(ii) it holds for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^*(\text{Hess } V)(x)) + \langle \mu(t, x), (\nabla V)(x) \rangle \leq \rho V(x). \quad (77)$$

Proof of Lemma 3.3. Throughout this proof let $\sigma_{i,j}: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}$, $i \in \{1, 2, \dots, d\}$, $j \in \{1, 2, \dots, m\}$, satisfy for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\sigma(t, x) = \begin{pmatrix} \sigma_{1,1}(t, x) & \sigma_{1,2}(t, x) & \dots & \sigma_{1,m}(t, x) \\ \sigma_{2,1}(t, x) & \sigma_{2,2}(t, x) & \dots & \sigma_{2,m}(t, x) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d,1}(t, x) & \sigma_{d,2}(t, x) & \dots & \sigma_{d,m}(t, x) \end{pmatrix} \in \mathbb{R}^{d \times m}. \quad (78)$$

Note that the chain rule, the fact that $\mathbb{R}^d \ni x \mapsto 1 + \|x\|^2 \in (0, \infty)$ is infinitely often differentiable, and the fact that $(0, \infty) \ni s \mapsto s^{\frac{p}{2}} \in (0, \infty)$ is infinitely often differentiable establish Item (i). It thus remains to prove Item (ii). For this, we observe for all $x \in \mathcal{O}$, $i, j \in \{1, 2, \dots, d\}$ that

$$(\nabla V)(x) = \frac{p}{2} (1 + \|x\|^2)^{\frac{p}{2}-1} \cdot (2x) = pV(x) \frac{x}{1+\|x\|^2} \quad (79)$$

and

$$\begin{aligned} \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)(x) &= \frac{\partial}{\partial x_i} \left[pV(x) \frac{x_j}{1+\|x\|^2} \right] = p \cdot \left(\frac{\partial V}{\partial x_i}\right)(x) \cdot \frac{x_j}{1+\|x\|^2} + pV(x) \cdot \left[\frac{\partial}{\partial x_i} \left(\frac{x_j}{1+\|x\|^2} \right) \right] \\ &= p^2 V(x) \frac{x_i x_j}{(1+\|x\|^2)^2} + pV(x) \frac{\delta_{ij}(1+\|x\|^2) - 2x_i x_j}{(1+\|x\|^2)^2} \\ &= p(p-2)V(x) \frac{x_i x_j}{(1+\|x\|^2)^2} + pV(x) \frac{\delta_{ij}}{1+\|x\|^2} \\ &= pV(x) \left[(p-2) \frac{x_i x_j}{(1+\|x\|^2)^2} + \frac{\delta_{ij}}{1+\|x\|^2} \right]. \end{aligned} \quad (80)$$

This yields for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\begin{aligned} &\frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^* (\text{Hess } V)(x)) + \langle \mu(t, x), (\nabla V)(x) \rangle \\ &= \frac{1}{2} \left[\sum_{k=1}^m \sum_{i,j=1}^d \sigma_{i,k}(t, x) \sigma_{j,k}(t, x) \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)(x) \right] + \left\langle \mu(t, x), pV(x) \frac{x}{1+\|x\|^2} \right\rangle \\ &= \frac{p}{2} \left[\left(\sum_{k=1}^m \sum_{i,j=1}^d \sigma_{i,k}(t, x) \sigma_{j,k}(t, x) \left((p-2) \frac{x_i x_j}{(1+\|x\|^2)^2} + \frac{\delta_{ij}}{1+\|x\|^2} \right) \right) + \frac{2\langle \mu(t, x), x \rangle}{1+\|x\|^2} \right] V(x) \\ &= \frac{p}{2} \left[\frac{(p-2)}{(1+\|x\|^2)^2} \left(\sum_{k=1}^m \left[\sum_{i=1}^d \sigma_{i,k}(t, x) x_i \right]^2 \right) + \frac{\|\sigma(t, x)\|^2}{1+\|x\|^2} + \frac{2\langle \mu(t, x), x \rangle}{1+\|x\|^2} \right] V(x). \end{aligned} \quad (81)$$

Next note that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\begin{aligned} \sum_{k=1}^m \left[\sum_{i=1}^d \sigma_{i,k}(t, x) x_i \right]^2 &\leq \sum_{k=1}^m \left(\sum_{i=1}^d |\sigma_{i,k}(t, x)|^2 \right) \left(\sum_{i=1}^d |x_i|^2 \right) = \|\sigma(t, x)\|^2 \|x\|^2 \\ &\leq c(1 + \|x\|^2) \|x\|^2 \leq c[1 + \|x\|^2]^2. \end{aligned} \quad (82)$$

Combining this with (81) shows that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\begin{aligned} &\frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^* (\text{Hess } V)(x)) + \langle \mu(t, x), (\nabla V)(x) \rangle \\ &\leq \frac{p}{2} [\max\{p-2, 0\}c + 3c] V(x) = \frac{pc}{2} \max\{p+1, 3\} V(x) = \rho V(x). \end{aligned} \quad (83)$$

This establishes Item (ii). The proof of Lemma 3.3 is thus completed. \square

3.2 Locality properties for solutions of SDEs

In this section we present two elementary results concerning the local behaviour of solutions to SDEs. These results, Lemmas 3.4 and 3.5 below, are used in the proof of Lemma 3.7 (see Section 3.4 below). Lemma 3.4 asserts, loosely speaking, that a particle whose movements are governed by a SDE with sufficiently regular coefficients is almost surely at rest when it finds itself in a region away from the supports of the coefficients.

Lemma 3.4. Let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, let $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\|\cdot\| : \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d \times m}$, let $\mu \in C([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in C([0, T] \times \mathbb{R}^d, \mathbb{R}^{d \times m})$ satisfy for all $r \in (0, \infty)$ that

$$\sup_{t \in [0, T]} \sup_{\substack{x, y \in \mathbb{R}^d, \\ x \neq y, \\ \|x\|, \|y\| \leq r}} \left[\frac{\|\mu(t, x) - \mu(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\|}{\|x - y\|} \right] < \infty, \quad (84)$$

let $\mathcal{O} \subseteq \mathbb{R}^d$ be an open set which satisfies $\text{supp}(\mu) \cup \text{supp}(\sigma) \subseteq [0, T] \times \mathcal{O}$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, and let $X : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ be an $(\mathbb{F}_t)_{t \in [0, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s. \quad (85)$$

Then

(i) it holds that $\left[(\mathbb{P}(X_0 \notin \mathcal{O}) = 1) \Rightarrow (\mathbb{P}(\forall t \in [0, T]: X_t = X_0) = 1) \right]$ and

(ii) it holds that $\left[(\mathbb{P}(X_0 \in \mathcal{O}) = 1) \Rightarrow (\mathbb{P}(\forall t \in [0, T]: X_t \in \overline{\mathcal{O}}) = 1) \right]$.

Proof of Lemma 3.4. We first prove Item (i). For this we assume that $\mathbb{P}(X_0 \notin \mathcal{O}) = 1$. Observe that this implies $\mathbb{P}(\forall t \in [0, T]: \|\mu(t, X_0)\| + \|\sigma(t, X_0)\| = 0) = 1$. Therefore, we obtain that

$$Y = ([0, T] \times \Omega \ni (t, \omega) \mapsto X_0(\omega) \in \mathbb{R}^d) \quad (86)$$

is an $(\mathbb{F}_t)_{t \in [0, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$\begin{aligned} Y_t &= X_0 = X_0 + \int_0^t 0 ds + \int_0^t 0 dW_s = X_0 + \int_0^t \mu(s, X_0) ds + \int_0^t \sigma(s, X_0) dW_s \\ &= X_0 + \int_0^t \mu(s, Y_s) ds + \int_0^t \sigma(s, Y_s) dW_s. \end{aligned} \quad (87)$$

Karatzas & Shreve [15, Theorem 5.2.5] and (84)–(86) hence assure that

$$\mathbb{P}(\forall t \in [0, T]: X_t = X_0) = \mathbb{P}(\forall t \in [0, T]: X_t = Y_t) = 1. \quad (88)$$

This establishes Item (i). Next we prove Item (ii). For this we assume that $\mathbb{P}(X_0 \in \mathcal{O}) = 1$ and let $\tau : \Omega \rightarrow [0, T]$ satisfy $\tau = \inf(\{t \in [0, T]: X_t \notin \overline{\mathcal{O}}\} \cup \{T\})$. Note that τ is an $(\mathbb{F}_t)_{t \in [0, T]}$ -stopping time. Let $Y : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ satisfy for all $t \in [0, T]$, $\omega \in \Omega$ that $Y_t(\omega) = X_{\min\{t, \tau(\omega)\}}(\omega)$. Observe that $Y : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ is an $(\mathbb{F}_t)_{t \in [0, T]}$ -adapted stochastic process with continuous sample paths. Moreover, note that for all $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$\begin{aligned} Y_t &= X_{\min\{t, \tau\}} = X_0 + \int_0^{\min\{t, \tau\}} \mu(s, X_s) ds + \int_0^{\min\{t, \tau\}} \sigma(s, X_s) dW_s \\ &= X_0 + \int_0^t \mathbb{1}_{\{0 < s \leq \tau\}} \mu(s, X_s) ds + \int_0^t \mathbb{1}_{\{0 < s \leq \tau\}} \sigma(s, X_s) dW_s. \end{aligned} \quad (89)$$

Combining this with the fact that for all $t \in [0, T]$ it holds that $\mathbb{1}_{\{t \leq \tau\}} X_t = \mathbb{1}_{\{t \leq \tau\}} Y_t$ and $\mathbb{1}_{\{\tau < t\}} [\|\mu(t, Y_t)\| + \|\sigma(t, Y_t)\|] = 0$ we obtain that for all $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$Y_t = X_0 + \int_0^t \mu(s, Y_s) ds + \int_0^t \sigma(s, Y_s) dW_s. \quad (90)$$

Karatzas & Shreve [15, Theorem 5.2.5], (84), and (85) hence demonstrate that

$$\mathbb{P}(\forall t \in [0, T]: X_t = Y_t) = 1. \quad (91)$$

This establishes Item (ii). The proof of Lemma 3.4 is thus completed. \square

The next result, Lemma 3.5 below, basically asserts that the solutions of SDEs coincide as long as the trajectories stay in a domain in which the drift and diffusion coefficients are the same.

Lemma 3.5. *Let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d \times m}$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be an open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}: \|x\| \leq r \text{ and } \{y \in \mathbb{R}^d: \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $\mathcal{C} \subseteq [0, T] \times \mathbb{R}^d$ be a closed set which satisfies $\mathcal{C} \subseteq [0, T] \times \mathcal{O}$, let $\mu_1, \mu_2 \in C([0, T] \times \mathcal{O}, \mathbb{R}^d)$, $\sigma_1, \sigma_2 \in C([0, T] \times \mathcal{O}, \mathbb{R}^{d \times m})$ satisfy for all $r \in (0, \infty)$ that $\mu_1|_{\mathcal{C}} = \mu_2|_{\mathcal{C}}$, $\sigma_1|_{\mathcal{C}} = \sigma_2|_{\mathcal{C}}$, and*

$$\sup \left(\left\{ \frac{\|\mu_1(t, x) - \mu_1(t, y)\| + \|\sigma_1(t, x) - \sigma_1(t, y)\|}{\|x - y\|} : t \in [0, T], x, y \in O_r, x \neq y \right\} \cup \{T\} \right) < \infty, \quad (92)$$

let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, let $X^{(i)} = (X_t^{(i)})_{t \in [0, T]}: [0, T] \times \Omega \rightarrow \mathcal{O}$, $i \in \{1, 2\}$, be $(\mathbb{F}_t)_{t \in [0, T]}$ -adapted stochastic processes with continuous sample paths which satisfy that for every $i \in \{1, 2\}$, $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$X_t^{(i)} = X_0^{(i)} + \int_0^t \mu_i(s, X_s^{(i)}) ds + \int_0^t \sigma_i(s, X_s^{(i)}) dW_s, \quad (93)$$

assume that $X_0^{(1)} = X_0^{(2)}$, and let $\tau: \Omega \rightarrow [0, T]$ satisfy $\tau = \inf(\{t \in [0, T]: (t, X_t^{(1)}) \notin \mathcal{C} \text{ or } (t, X_t^{(2)}) \notin \mathcal{C}\} \cup \{T\})$. Then it holds that

$$\mathbb{P}(\forall t \in [0, T]: \mathbb{1}_{\{t \leq \tau\}} \|X_t^{(1)} - X_t^{(2)}\| = 0) = 1. \quad (94)$$

Proof of Lemma 3.5. Throughout this proof let $\rho_n: \Omega \rightarrow [0, T]$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$ that $\rho_n = \inf(\{t \in [0, T]: X_t^{(1)} \in \mathcal{O} \setminus O_n \text{ or } X_t^{(2)} \in \mathcal{O} \setminus O_n\} \cup \{T\})$ and let $L_n \in [0, \infty)$, $n \in \mathbb{N}$, be real numbers which satisfy for all $t \in [0, T]$, $x, y \in O_n$ that

$$\|\mu_1(t, x) - \mu_1(t, y)\| + \|\sigma_1(t, x) - \sigma_1(t, y)\| \leq L_n \|x - y\|. \quad (95)$$

Observe that for all $n \in \mathbb{N}$ it holds that τ and ρ_n are $(\mathbb{F}_t)_{t \in [0, T]}$ -stopping times. Moreover, note that for every $K \subseteq \mathcal{O}$ compact there exists $n \in \mathbb{N}$ such that $K \subseteq O_n$. This and the fact that $X^{(1)}$ and $X^{(2)}$ have continuous sample paths ensure that for all $\omega \in \Omega$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq n$ it holds that $\rho_k(\omega) = T$. Next note that the assumption that $X^{(1)}$ and $X^{(2)}$ have continuous sample paths and the fact that O_n , $n \in \mathbb{N}$, are compact imply that for all $n \in \mathbb{N}$, $\omega \in \{\rho_n > 0\}$, $i \in \{1, 2\}$ it holds that $X_{\rho_n(\omega)}^{(i)}(\omega) \in O_n$. Combining this with the assumption that $X_0^{(1)} = X_0^{(2)}$ assures that for all $n \in \mathbb{N}$, $t \in [0, T]$ it holds that

$$\|X_{\min\{t, \tau, \rho_n\}}^{(1)} - X_{\min\{t, \tau, \rho_n\}}^{(2)}\| \leq 2n. \quad (96)$$

This ensures for every $n \in \mathbb{N}$ that

$$\sup_{t \in [0, T]} \left(\mathbb{E} \left[\left\| X_{\min\{t, \tau, \rho_n\}}^{(1)} - X_{\min\{t, \tau, \rho_n\}}^{(2)} \right\|^2 \right] \right) < \infty. \quad (97)$$

Next note that the fact that for all $s \in (0, T]$ it holds that $\mathbb{1}_{\{s \leq \tau\}} [\|\mu_1(s, X_s^{(2)}) - \mu_2(s, X_s^{(2)})\| + \|\sigma_1(s, X_s^{(2)}) - \sigma_2(s, X_s^{(2)})\|] = 0$, the assumption that $X_0^{(1)} = X_0^{(2)}$, and (93) ensure that for all $n \in \mathbb{N}$, $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$\begin{aligned}
X_{\min\{t, \tau, \rho_n\}}^{(1)} - X_{\min\{t, \tau, \rho_n\}}^{(2)} &= \int_0^{\min\{t, \tau, \rho_n\}} [\mu_1(s, X_s^{(1)}) - \mu_2(s, X_s^{(2)})] ds \\
&\quad + \int_0^{\min\{t, \tau, \rho_n\}} [\sigma_1(s, X_s^{(1)}) - \sigma_2(s, X_s^{(2)})] dW_s \\
&= \int_0^t \mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} [\mu_1(s, X_s^{(1)}) - \mu_2(s, X_s^{(2)})] ds \\
&\quad + \int_0^t \mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} [\sigma_1(s, X_s^{(1)}) - \sigma_2(s, X_s^{(2)})] dW_s \\
&= \int_0^t \mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} [\mu_1(s, X_s^{(1)}) - \mu_1(s, X_s^{(2)})] ds \\
&\quad + \int_0^t \mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} [\sigma_1(s, X_s^{(1)}) - \sigma_1(s, X_s^{(2)})] dW_s.
\end{aligned} \tag{98}$$

This implies that for all $n \in \mathbb{N}$, $t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$\begin{aligned}
&X_{\min\{t, \tau, \rho_n\}}^{(1)} - X_{\min\{t, \tau, \rho_n\}}^{(2)} \\
&= \int_0^t \mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} [\mu_1(s, X_{\min\{s, \tau, \rho_n\}}^{(1)}) - \mu_1(s, X_{\min\{s, \tau, \rho_n\}}^{(2)})] ds \\
&\quad + \int_0^t \mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} [\sigma_1(s, X_{\min\{s, \tau, \rho_n\}}^{(1)}) - \sigma_1(s, X_{\min\{s, \tau, \rho_n\}}^{(2)})] dW_s.
\end{aligned} \tag{99}$$

Minkowski's inequality, Itô's isometry, and (95)–(97) hence yield for all $n \in \mathbb{N}$, $t \in [0, T]$ that

$$\begin{aligned}
&\mathbb{E} \left[\left\| X_{\min\{t, \tau, \rho_n\}}^{(1)} - X_{\min\{t, \tau, \rho_n\}}^{(2)} \right\|^2 \right]^{1/2} \\
&\leq \int_0^t \mathbb{E} \left[\mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} \left\| \mu_1(s, X_{\min\{s, \tau, \rho_n\}}^{(1)}) - \mu_1(s, X_{\min\{s, \tau, \rho_n\}}^{(2)}) \right\|^2 \right]^{1/2} ds \\
&\quad + \mathbb{E} \left[\left\| \int_0^t \mathbb{1}_{\{0 < s \leq \min\{\tau, \rho_n\}\}} [\sigma_1(s, X_{\min\{s, \tau, \rho_n\}}^{(1)}) - \sigma_1(s, X_{\min\{s, \tau, \rho_n\}}^{(2)})] dW_s \right\|^2 \right]^{1/2} \\
&\leq \int_0^t \mathbb{E} \left[\left\| \mu_1(s, X_{\min\{s, \tau, \rho_n\}}^{(1)}) - \mu_1(s, X_{\min\{s, \tau, \rho_n\}}^{(2)}) \right\|^2 \right]^{1/2} ds \\
&\quad + \left[\int_0^t \mathbb{E} \left[\left\| \sigma_1(s, X_{\min\{s, \tau, \rho_n\}}^{(1)}) - \sigma_1(s, X_{\min\{s, \tau, \rho_n\}}^{(2)}) \right\|^2 \right] ds \right]^{1/2} \\
&\leq L_n \int_0^t \mathbb{E} \left[\left\| X_{\min\{s, \tau, \rho_n\}}^{(1)} - X_{\min\{s, \tau, \rho_n\}}^{(2)} \right\|^2 \right]^{1/2} ds \\
&\quad + L_n \left[\int_0^t \mathbb{E} \left[\left\| X_{\min\{s, \tau, \rho_n\}}^{(1)} - X_{\min\{s, \tau, \rho_n\}}^{(2)} \right\|^2 \right] ds \right]^{1/2}.
\end{aligned} \tag{100}$$

The fact that for all $a, b \in [0, \infty)$ it holds that $(a + b)^2 \leq 2a^2 + 2b^2$ and Hölder's inequality hence demonstrate for all $n \in \mathbb{N}$, $t \in [0, T]$ that

$$\mathbb{E} \left[\left\| X_{\min\{t, \tau, \rho_n\}}^{(1)} - X_{\min\{t, \tau, \rho_n\}}^{(2)} \right\|^2 \right] \leq 2(L_n)^2(T + 1) \int_0^t \mathbb{E} \left[\left\| X_{\min\{s, \tau, \rho_n\}}^{(1)} - X_{\min\{s, \tau, \rho_n\}}^{(2)} \right\|^2 \right] ds. \tag{101}$$

Combining this with Gronwall's inequality and (97) implies for all $n \in \mathbb{N}$, $t \in [0, T]$ that

$$\mathbb{E} \left[\left\| X_{\min\{t, \tau, \rho_n\}}^{(1)} - X_{\min\{t, \tau, \rho_n\}}^{(2)} \right\|^2 \right] = 0. \quad (102)$$

The fact that $X^{(1)}$ and $X^{(2)}$ have continuous sample paths hence ensures for all $n \in \mathbb{N}$ that

$$\mathbb{P} \left(\forall t \in [0, T]: \mathbb{1}_{\{t \leq \min\{\tau, \rho_n\}\}} \|X_t^{(1)} - X_t^{(2)}\| = 0 \right) = 1. \quad (103)$$

Therefore we obtain that

$$\mathbb{P} \left(\forall n \in \mathbb{N} \forall t \in [0, T]: \mathbb{1}_{\{t \leq \min\{\tau, \rho_n\}\}} \|X_t^{(1)} - X_t^{(2)}\| = 0 \right) = 1. \quad (104)$$

This implies that

$$\mathbb{P} \left(\forall t \in [0, T]: \mathbb{1}_{\{t \leq \tau\}} \|X_t^{(1)} - X_t^{(2)}\| = 0 \right) = 1. \quad (105)$$

This establishes (94). The proof of Lemma 3.5 is thus completed. \square

3.3 Continuity properties for solutions of SDEs

The well-known Lemma 3.6 below (cf. also Stroock [21, Theorem I.2.2]) estimates the difference between two solutions to the same SDE that start at different times and different places. Lemma 3.6 is a crucial ingredient in the proof of Lemma 3.7, where it is used to show that the solution to an auxiliary SDE evaluated at a certain time is stochastically continuous as a function of the initial values.

Lemma 3.6. *Let $d, m \in \mathbb{N}$, $L, T \in (0, \infty)$, let $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\|\cdot\| : \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d \times m}$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, let $\mu \in C([0, T] \times \mathcal{O}, \mathbb{R}^d)$, $\sigma \in C([0, T] \times \mathcal{O}, \mathbb{R}^{d \times m})$ be compactly supported functions which satisfy for all $t \in [0, T]$, $x, y \in \mathcal{O}$ that*

$$\|\mu(t, x) - \mu(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq L \|x - y\|, \quad (106)$$

let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, and for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t, x} = (X_s^{t, x})_{s \in [t, T]} : [t, T] \times \Omega \rightarrow \mathcal{O}$ be an $(\mathbb{F}_s)_{s \in [t, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$X_s^{t, x} = x + \int_t^s \mu(r, X_r^{t, x}) dr + \int_t^s \sigma(r, X_r^{t, x}) dW_r. \quad (107)$$

Then it holds for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $s \in [\mathfrak{t}, T]$, $x, \mathfrak{x} \in \mathcal{O}$ that

$$\begin{aligned} & \mathbb{E} \left[\left\| X_s^{t, x} - X_s^{\mathfrak{t}, \mathfrak{x}} \right\|^2 \right] \\ & \leq 9 \left[\|x - \mathfrak{x}\| + |t - \mathfrak{t}|^{1/2} \right]^2 \left[1 + \sqrt{T} \sup_{r \in [0, T]} \|\mu(r, \mathfrak{x})\| + \sup_{r \in [0, T]} \|\sigma(r, \mathfrak{x})\| \right]^2 \exp(6L^2 T(T+1)). \end{aligned} \quad (108)$$

Proof of Lemma 3.6. Throughout this proof let $\mathfrak{m} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mathfrak{s} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ satisfy for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$\mathfrak{m}(t, x) = \begin{cases} \mu(t, x) & : x \in \mathcal{O} \\ 0 & : x \in \mathbb{R}^d \setminus \mathcal{O} \end{cases} \quad \text{and} \quad \mathfrak{s}(t, x) = \begin{cases} \sigma(t, x) & : x \in \mathcal{O} \\ 0 & : x \in \mathbb{R}^d \setminus \mathcal{O}. \end{cases} \quad (109)$$

Observe that (106) ensures that $\mathbf{m}: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mathbf{s}: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ are compactly supported continuous functions which satisfy for all $t \in [0, T]$, $x, y \in \mathbb{R}^d$ that

$$\|\mathbf{m}(t, x) - \mathbf{m}(t, y)\| + \|\mathbf{s}(t, x) - \mathbf{s}(t, y)\| \leq L \|x - y\|. \quad (110)$$

Karatzas & Shreve [15, Theorem 5.2.9] hence guarantees for every $t \in [0, T]$, $x \in \mathcal{O}$ that there exists an $(\mathbb{F}_s)_{s \in [t, T]}$ -adapted stochastic process $\mathfrak{X}^{t, x} = (\mathfrak{X}_s^{t, x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathbb{R}^d$ with continuous sample paths

(A) which satisfies that $\sup_{s \in [t, T]} \mathbb{E}[\|\mathfrak{X}_s^{t, x}\|^2] < \infty$ and

(B) which satisfies that for all $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$\mathfrak{X}_s^{t, x} = x + \int_t^s \mathbf{m}(r, \mathfrak{X}_r^{t, x}) dr + \int_t^s \mathbf{s}(r, \mathfrak{X}_r^{t, x}) dW_r. \quad (111)$$

This, Karatzas & Shreve [15, Theorem 5.2.5], and (107) ensure that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\mathbb{P}(\forall s \in [t, T]: X_s^{t, x} = \mathfrak{X}_s^{t, x}) = 1. \quad (112)$$

Combining this with the fact that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that $\sup_{s \in [t, T]} \mathbb{E}[\|\mathfrak{X}_s^{t, x}\|^2] < \infty$ implies that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\sup_{s \in [t, T]} \mathbb{E}[\|X_s^{t, x}\|^2] < \infty. \quad (113)$$

Next note that (107) ensures that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $s \in [\mathfrak{t}, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds \mathbb{P} -a.s. that

$$X_s^{t, x} - X_s^{\mathfrak{t}, \mathfrak{x}} = X_{\mathfrak{t}}^{t, x} - \mathfrak{x} + \int_{\mathfrak{t}}^s (\mu(r, X_r^{t, x}) - \mu(r, X_r^{\mathfrak{t}, \mathfrak{x}})) dr + \int_{\mathfrak{t}}^s (\sigma(r, X_r^{t, x}) - \sigma(r, X_r^{\mathfrak{t}, \mathfrak{x}})) dW_r. \quad (114)$$

Minkowski's inequality hence yields that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $s \in [\mathfrak{t}, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds that

$$\begin{aligned} \left| \mathbb{E}[\|X_s^{t, x} - X_s^{\mathfrak{t}, \mathfrak{x}}\|^2] \right|^{1/2} &\leq \left| \mathbb{E}[\|X_{\mathfrak{t}}^{t, x} - \mathfrak{x}\|^2] \right|^{1/2} + \int_{\mathfrak{t}}^s \left| \mathbb{E}[\|\mu(r, X_r^{t, x}) - \mu(r, X_r^{\mathfrak{t}, \mathfrak{x}})\|^2] \right|^{1/2} dr \\ &\quad + \left| \mathbb{E} \left[\left\| \int_{\mathfrak{t}}^s (\sigma(r, X_r^{t, x}) - \sigma(r, X_r^{\mathfrak{t}, \mathfrak{x}})) dW_r \right\|^2 \right] \right|^{1/2}. \end{aligned} \quad (115)$$

Itô's isometry and (106) therefore ensure that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $s \in [\mathfrak{t}, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds that

$$\begin{aligned} \left| \mathbb{E}[\|X_s^{t, x} - X_s^{\mathfrak{t}, \mathfrak{x}}\|^2] \right|^{1/2} &\leq \left| \mathbb{E}[\|X_{\mathfrak{t}}^{t, x} - \mathfrak{x}\|^2] \right|^{1/2} + L \int_{\mathfrak{t}}^s \left| \mathbb{E}[\|X_r^{t, x} - X_r^{\mathfrak{t}, \mathfrak{x}}\|^2] \right|^{1/2} dr \\ &\quad + \left| \int_{\mathfrak{t}}^s \mathbb{E}[\|\sigma(r, X_r^{t, x}) - \sigma(r, X_r^{\mathfrak{t}, \mathfrak{x}})\|^2] dr \right|^{1/2}. \end{aligned} \quad (116)$$

This and (106) imply for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $s \in [\mathfrak{t}, T]$, $x, \mathfrak{x} \in \mathcal{O}$ that

$$\begin{aligned} &\left| \mathbb{E}[\|X_s^{t, x} - X_s^{\mathfrak{t}, \mathfrak{x}}\|^2] \right|^{1/2} \\ &\leq \left| \mathbb{E}[\|X_{\mathfrak{t}}^{t, x} - \mathfrak{x}\|^2] \right|^{1/2} + L \int_{\mathfrak{t}}^s \left| \mathbb{E}[\|X_r^{t, x} - X_r^{\mathfrak{t}, \mathfrak{x}}\|^2] \right|^{1/2} dr + L \left| \int_{\mathfrak{t}}^s \mathbb{E}[\|X_r^{t, x} - X_r^{\mathfrak{t}, \mathfrak{x}}\|^2] dr \right|^{1/2}. \end{aligned} \quad (117)$$

The fact that for all $a, b, c \in [0, \infty)$ it holds that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and Hölder's inequality therefore ensure that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $s \in [\mathfrak{t}, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds that

$$\begin{aligned} & \mathbb{E} \left[\|X_s^{t,x} - X_s^{\mathfrak{t},\mathfrak{x}}\|^2 \right] \\ & \leq 3\mathbb{E} \left[\|X_t^{t,x} - \mathfrak{x}\|^2 \right] + 3L^2T \int_t^s \mathbb{E} \left[\|X_r^{t,x} - X_r^{\mathfrak{t},\mathfrak{x}}\|^2 \right] dr + 3L^2 \int_t^s \mathbb{E} \left[\|X_r^{t,x} - X_r^{\mathfrak{t},\mathfrak{x}}\|^2 \right] dr \\ & = 3\mathbb{E} \left[\|X_t^{t,x} - \mathfrak{x}\|^2 \right] + 3L^2(T+1) \int_t^s \mathbb{E} \left[\|X_r^{t,x} - X_r^{\mathfrak{t},\mathfrak{x}}\|^2 \right] dr. \end{aligned} \quad (118)$$

Gronwall's inequality and (113) hence imply for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $s \in [\mathfrak{t}, T]$, $x, \mathfrak{x} \in \mathcal{O}$ that

$$\mathbb{E} \left[\|X_s^{t,x} - X_s^{\mathfrak{t},\mathfrak{x}}\|^2 \right] \leq 3\mathbb{E} \left[\|X_t^{t,x} - \mathfrak{x}\|^2 \right] \exp(3L^2T(T+1)). \quad (119)$$

In the next step we observe that (107) guarantees that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds \mathbb{P} -a.s. that

$$X_{\mathfrak{t}}^{t,x} - \mathfrak{x} = x - \mathfrak{x} + \int_t^{\mathfrak{t}} \mu(r, X_r^{t,x}) dr + \int_t^{\mathfrak{t}} \sigma(r, X_r^{t,x}) dW_r. \quad (120)$$

Minkowski's inequality hence demonstrates that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds that

$$\left| \mathbb{E} \left[\|X_{\mathfrak{t}}^{t,x} - \mathfrak{x}\|^2 \right] \right|^{1/2} \leq \|x - \mathfrak{x}\| + \int_t^{\mathfrak{t}} \left| \mathbb{E} \left[\|\mu(r, X_r^{t,x})\|^2 \right] \right|^{1/2} dr + \left| \mathbb{E} \left[\left\| \int_t^{\mathfrak{t}} \sigma(r, X_r^{t,x}) dW_r \right\|^2 \right] \right|^{1/2}. \quad (121)$$

Itô's isometry therefore implies that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds that

$$\left| \mathbb{E} \left[\|X_{\mathfrak{t}}^{t,x} - \mathfrak{x}\|^2 \right] \right|^{1/2} \leq \|x - \mathfrak{x}\| + \int_t^{\mathfrak{t}} \left| \mathbb{E} \left[\|\mu(r, X_r^{t,x})\|^2 \right] \right|^{1/2} dr + \left| \int_t^{\mathfrak{t}} \mathbb{E} \left[\|\sigma(r, X_r^{t,x})\|^2 \right] dr \right|^{1/2}. \quad (122)$$

This, Minkowski's inequality, and (106) yield that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds that

$$\begin{aligned} \left| \mathbb{E} \left[\|X_{\mathfrak{t}}^{t,x} - \mathfrak{x}\|^2 \right] \right|^{1/2} & \leq \|x - \mathfrak{x}\| + \int_t^{\mathfrak{t}} \|\mu(r, \mathfrak{x})\| dr + L \int_t^{\mathfrak{t}} \left| \mathbb{E} \left[\|X_r^{t,x} - \mathfrak{x}\|^2 \right] \right|^{1/2} dr \\ & \quad + \left| \int_t^{\mathfrak{t}} \|\sigma(r, \mathfrak{x})\|^2 dr \right|^{1/2} + L \left| \int_t^{\mathfrak{t}} \mathbb{E} \left[\|X_r^{t,x} - \mathfrak{x}\|^2 \right] dr \right|^{1/2}. \end{aligned} \quad (123)$$

The fact that for all $a, b, c \in [0, \infty)$ it holds that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ hence implies for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $x, \mathfrak{x} \in \mathcal{O}$ that

$$\begin{aligned} & \mathbb{E} \left[\|X_{\mathfrak{t}}^{t,x} - \mathfrak{x}\|^2 \right] \\ & \leq 3 \left[\|x - \mathfrak{x}\| + \int_t^{\mathfrak{t}} \|\mu(r, \mathfrak{x})\| dr + \left| \int_t^{\mathfrak{t}} \|\sigma(r, \mathfrak{x})\|^2 dr \right|^{1/2} \right]^2 + 3L^2(T+1) \int_t^{\mathfrak{t}} \mathbb{E} \left[\|X_r^{t,x} - \mathfrak{x}\|^2 \right] dr. \end{aligned} \quad (124)$$

This demonstrates for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $x, \mathfrak{x} \in \mathcal{O}$ that

$$\begin{aligned} \mathbb{E} \left[\|X_{\mathfrak{t}}^{t,x} - \mathfrak{x}\|^2 \right] & \leq 3 \left[\|x - \mathfrak{x}\| + |\mathfrak{t} - t| \sup_{r \in [0, T]} \|\mu(r, \mathfrak{x})\| + |\mathfrak{t} - t|^{1/2} \sup_{r \in [0, T]} \|\sigma(r, \mathfrak{x})\| \right]^2 \\ & \quad + 3L^2(T+1) \int_t^{\mathfrak{t}} \mathbb{E} \left[\|X_r^{t,x} - \mathfrak{x}\|^2 \right] dr. \end{aligned} \quad (125)$$

Hence, we obtain for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $x, \mathfrak{x} \in \mathcal{O}$ that

$$\begin{aligned} \mathbb{E} \left[\|X_{\mathfrak{t}}^{t,x} - \mathfrak{x}\|^2 \right] &\leq 3 \left[\|x - \mathfrak{x}\| + |t - \mathfrak{t}|^{1/2} \right]^2 \left[1 + \sqrt{T} \sup_{r \in [0, T]} \|\mu(r, \mathfrak{x})\| + \sup_{r \in [0, T]} \|\sigma(r, \mathfrak{x})\| \right]^2 \\ &\quad + 3L^2(T+1) \int_t^{\mathfrak{t}} \mathbb{E} \left[\|X_r^{t,x} - \mathfrak{x}\|^2 \right] dr. \end{aligned} \quad (126)$$

Gronwall's inequality and (113) hence ensure that for all $t \in [0, T]$, $\mathfrak{t} \in [t, T]$, $x, \mathfrak{x} \in \mathcal{O}$ it holds that

$$\begin{aligned} \mathbb{E} \left[\|X_{\mathfrak{t}}^{t,x} - \mathfrak{x}\|^2 \right] &\leq 3 \left[\|x - \mathfrak{x}\| + |t - \mathfrak{t}|^{1/2} \right]^2 \left[1 + \sqrt{T} \sup_{r \in [0, T]} \|\mu(r, \mathfrak{x})\| + \sup_{r \in [0, T]} \|\sigma(r, \mathfrak{x})\| \right]^2 \exp(3L^2T(T+1)). \end{aligned} \quad (127)$$

Combining this with (119) demonstrates (108). The proof of Lemma 3.6 is thus completed. \square

3.4 Existence and uniqueness properties for solutions of SFPEs associated with SDEs

In this section we provide the announced application of Theorem 2.9 (see Theorem 3.8 below). The next essentially well-known result, Lemma 3.7 below (cf., for example, Liu & Röckner [17, Proposition 3.2.1]), ascertains that the stochastic continuity hypothesis of Theorem 2.9 is satisfied in the setting of Theorem 3.8.

Lemma 3.7. *Let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the standard scalar product on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d \times m}$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}: \|x\| \leq r \text{ and } \{y \in \mathbb{R}^d: \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $\mu \in C([0, T] \times \mathcal{O}, \mathbb{R}^d)$, $\sigma \in C([0, T] \times \mathcal{O}, \mathbb{R}^{d \times m})$ satisfy for all $r \in (0, \infty)$ that*

$$\sup \left(\left\{ \frac{\|\mu(t, x) - \mu(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\|}{\|x - y\|} : t \in [0, T], x, y \in O_r, x \neq y \right\} \cup \{0\} \right) < \infty, \quad (128)$$

let $V \in C^{1,2}([0, T] \times \mathcal{O}, (0, \infty))$ satisfy for all $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\left(\frac{\partial V}{\partial t} \right)(t, x) + \frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^* (\text{Hess}_x V)(t, x)) + \langle \mu(t, x), (\nabla_x V)(t, x) \rangle \leq 0, \quad (129)$$

assume that $\sup_{r \in (0, \infty)} [\inf_{t \in [0, T]} \inf_{x \in \mathcal{O} \setminus O_r} V(t, x)] = \infty$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, and for every $t \in [0, T]$, $x \in \mathcal{O}$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathcal{O}$ be an $(\mathbb{F}_s)_{s \in [t, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$X_s^{t,x} = x + \int_t^s \mu(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r. \quad (130)$$

Then it holds for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ and all $(t_n, x_n) \in [0, T] \times \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n \rightarrow \infty} [|t_n - t_0| + \|x_n - x_0\|] = 0$ that

$$\limsup_{n \rightarrow \infty} \left[\mathbb{P} \left(\|X_{\max\{s, t_n\}}^{t_n, x_n} - X_{\max\{s, t_0\}}^{t_0, x_0}\| \geq \varepsilon \right) \right] = 0. \quad (131)$$

Proof of Lemma 3.7. Throughout this proof let $(\mathbf{t}_n, \mathbf{x}_n) \in [0, T] \times \mathcal{O}$, $n \in \mathbb{N}_0$, satisfy $\limsup_{n \rightarrow \infty} [\|\mathbf{t}_n - \mathbf{t}_0\| + \|\mathbf{x}_n - \mathbf{x}_0\|] = 0$. Note that for the proof of Lemma 3.7 it is sufficient to demonstrate that for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ it holds that

$$\limsup_{n \rightarrow \infty} \left[\mathbb{P} \left(\left\| X_{\max\{s, \mathbf{t}_n\}}^{\mathbf{t}_n, \mathbf{x}_n} - X_{\max\{s, \mathbf{t}_0\}}^{\mathbf{t}_0, \mathbf{x}_0} \right\| \geq \varepsilon \right) \right] = 0. \quad (132)$$

Next note that the assumption that it holds that $\sup_{r \in (0, \infty)} \inf_{t \in [0, T]} \inf_{x \in \mathcal{O} \setminus O_r} V(t, x) = \infty$ ensures that for every $n \in \mathbb{N}$ there exists $r \in (0, \infty)$ such that $\inf_{t \in [0, T]} \inf_{x \in \mathcal{O} \setminus O_r} V(t, x) > n$. This yields that for every $n \in \mathbb{N}$ there exists $r \in (0, \infty)$ such that $\{V \leq n\} \subseteq [0, T] \times O_r$. Hence, we obtain for every $n \in \mathbb{N}$ that $\{V \leq n\}$ is a bounded set. Combining this with the fact that $V: [0, T] \times \mathcal{O} \rightarrow (0, \infty)$ is continuous demonstrates that for every $n \in \mathbb{N}$ it holds that $\{V \leq n\}$ is a compact set. Lang [16, Theorem II.3.7] therefore ensures that there exist $\varphi_n \in C_c^\infty([0, T] \times \mathcal{O}, \mathbb{R})$, $n \in \mathbb{N}$, which satisfy for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\mathbb{1}_{\{V \leq n\}}(t, x) \leq \varphi_n(t, x) \leq \mathbb{1}_{\{V < n+1\}}(t, x). \quad (133)$$

Next let $\mathbf{m}_n: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $n \in \mathbb{N}$, and $\mathbf{s}_n: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$\mathbf{m}_n(t, x) = \begin{cases} \varphi_n(t, x) \mu(t, x) & : x \in \mathcal{O} \\ 0 & : x \in \mathbb{R}^d \setminus \mathcal{O} \end{cases} \quad \text{and} \quad \mathbf{s}_n(t, x) = \begin{cases} \varphi_n(t, x) \sigma(t, x) & : x \in \mathcal{O} \\ 0 & : x \in \mathbb{R}^d \setminus \mathcal{O}. \end{cases} \quad (134)$$

This, (128), and (133) assure that $\mathbf{m}_n: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $n \in \mathbb{N}$, and $\mathbf{s}_n: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$, $n \in \mathbb{N}$, are compactly supported continuous functions which satisfy that

(A) for all $n \in \mathbb{N}$ it holds that

$$\sup_{t \in [0, T]} \sup_{\substack{x, y \in \mathbb{R}^d \\ x \neq y}} \left[\frac{\|\mathbf{m}_n(t, x) - \mathbf{m}_n(t, y)\| + \|\mathbf{s}_n(t, x) - \mathbf{s}_n(t, y)\|}{\|x - y\|} \right] < \infty, \quad (135)$$

(B) for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\left[\|\mathbf{m}_n(t, x) - \mu(t, x)\| + \|\mathbf{s}_n(t, x) - \sigma(t, x)\| \right] \mathbb{1}_{\{V \leq n\}}(t, x) = 0, \quad (136)$$

and

(C) for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ it holds that

$$\left[\|\mathbf{m}_n(t, x)\| + \|\mathbf{s}_n(t, x)\| \right] \mathbb{1}_{\{V \geq n+1\}}(t, x) = 0. \quad (137)$$

Note that Karatzas & Shreve [15, Theorem 5.2.9] (cf. also Gyöngy & Krylov [8, Corollary 2.6] and Liu & Röckner [17, Theorem 3.1.1]) and Item (A) yield that for every $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ there exists an $(\mathbb{F}_s)_{s \in [t, T]}$ -adapted stochastic process with continuous sample paths $\mathbb{X}^{n, t, x} = (\mathbb{X}_s^{n, t, x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathbb{R}^d$ which satisfies that for all $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$\mathbb{X}_s^{n, t, x} = x + \int_t^s \mathbf{m}_n(r, \mathbb{X}_r^{n, t, x}) dr + \int_t^s \mathbf{s}_n(r, \mathbb{X}_r^{n, t, x}) dW_r. \quad (138)$$

Moreover, note that Item (C) ensures for all $n \in \mathbb{N}$ that $\text{supp}(\mathbf{m}_n) \cup \text{supp}(\mathbf{s}_n) \subseteq \{V \leq n+1\}$. The fact that for every $n \in \mathbb{N}$ there exists $r \in (0, \infty)$ such that $\{V \leq n\} \subseteq [0, T] \times O_r$ hence implies that for every $n \in \mathbb{N}$ there exists $r \in (0, \infty)$ such that $\text{supp}(\mathbf{m}_n) \cup \text{supp}(\mathbf{s}_n) \subseteq [0, T] \times O_r$. Furthermore,

observe that Item (i) of Lemma 3.4 ensures that for all $n \in \mathbb{N}$, $r \in (0, \infty)$, $m \in \mathbb{N} \cap (r, \infty)$, $t \in [0, T]$, $x \in \mathcal{O} \setminus \{y \in \mathcal{O} : (\exists z \in O_r : \|y - z\| < 1/m)\}$ with $\text{supp}(\mathbf{m}_n) \cup \text{supp}(\mathbf{s}_n) \subseteq [0, T] \times O_r$ it holds that $\mathbb{P}(\forall s \in [t, T] : \mathbb{X}_s^{n,t,x} = x) = 1$. Combining this with the fact that for all $r \in (0, \infty)$ it holds that $O_r = \bigcap_{m \in \mathbb{N} \cap (r, \infty)} \{y \in \mathbb{R}^d : (\exists x \in O_r : \|x - y\| < 1/m)\}$ implies that for all $n \in \mathbb{N}$, $r \in (0, \infty)$, $t \in [0, T]$, $x \in \mathcal{O} \setminus O_r$ with $\text{supp}(\mathbf{m}_n) \cup \text{supp}(\mathbf{s}_n) \subseteq [0, T] \times O_r$ it holds that

$$\mathbb{P}(\forall s \in [t, T] : \mathbb{X}_s^{n,t,x} = x) = 1. \quad (139)$$

Next we observe that Item (ii) of Lemma 3.4 ensures that for all $n \in \mathbb{N}$, $r \in (0, \infty)$, $m \in \mathbb{N} \cap (r, \infty)$, $t \in [0, T]$, $x \in \{y \in \mathcal{O} : (\exists z \in O_r : \|y - z\| < 1/m)\}$ with $\text{supp}(\mathbf{m}_n) \cup \text{supp}(\mathbf{s}_n) \subseteq [0, T] \times O_r$ it holds that $\mathbb{P}(\forall s \in [t, T] : (\exists y \in O_r : \|\mathbb{X}_s^{n,t,x} - y\| \leq 1/m)) = 1$. This yields that for all $n \in \mathbb{N}$, $r \in (0, \infty)$, $t \in [0, T]$, $x \in O_r$ with $\text{supp}(\mathbf{m}_n) \cup \text{supp}(\mathbf{s}_n) \subseteq [0, T] \times O_r$ it holds that

$$\mathbb{P}(\forall s \in [t, T] : \mathbb{X}_s^{n,t,x} \in O_r) = 1. \quad (140)$$

The fact that for every $n \in \mathbb{N}$ there exists $r \in (0, \infty)$ such that $\text{supp}(\mathbf{m}_n) \cup \text{supp}(\mathbf{s}_n) \subseteq [0, T] \times O_r$ and (139) hence demonstrate that for every $n \in \mathbb{N}$ there exists $r \in (0, \infty)$ such that

(I) it holds for all $t \in [0, T]$, $x \in O_r$ that $\mathbb{P}(\forall s \in [t, T] : \mathbb{X}_s^{n,t,x} \in O_r) = 1$ and

(II) it holds for all $t \in [0, T]$, $x \in \mathcal{O} \setminus O_r$ that $\mathbb{P}(\forall s \in [t, T] : \mathbb{X}_s^{n,t,x} = x) = 1$.

Therefore, we obtain that for every $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ there exists an $(\mathbb{F}_s)_{s \in [t, T]}$ -adapted stochastic process with continuous sample paths $\mathfrak{X}^{n,t,x} = (\mathfrak{X}_s^{n,t,x})_{s \in [t, T]} : [t, T] \times \Omega \rightarrow \mathcal{O}$ which satisfies that for all $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$\mathfrak{X}_s^{n,t,x} = x + \int_t^s \mathbf{m}_n(r, \mathfrak{X}_r^{n,t,x}) dr + \int_t^s \mathbf{s}_n(r, \mathfrak{X}_r^{n,t,x}) dW_r. \quad (141)$$

In the next step let $\tau^{n,t,x} : \Omega \rightarrow [t, T]$, $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$, satisfy for every $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$, $\omega \in \Omega$ that $\tau^{n,t,x}(\omega) = \inf(\{s \in [t, T] : \max\{V(s, \mathfrak{X}_s^{n,t,x}(\omega)), V(s, X_s^{t,x}(\omega))\} > n\} \cup \{T\})$. Note that for every $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ it holds that $\tau^{n,t,x} : \Omega \rightarrow [t, T]$ is an $(\mathbb{F}_s)_{s \in [t, T]}$ -stopping time. Next observe that the fact that for every $n \in \mathbb{N}$ it holds that $\{V \leq n\}$ is a compact set, Item (B), and Lemma 3.5 (with $T = T - t$, $\mathcal{C} = \{(s, y) \in [0, T - t] \times \mathcal{O} : V(t + s, y) \leq n\}$, $\mu_1 = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto \mu(t + s, y) \in \mathbb{R}^d)$, $\mu_2 = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto \mathbf{m}_n(t + s, y) \in \mathbb{R}^d)$, $\sigma_1 = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto \sigma(t + s, y) \in \mathbb{R}^{d \times m})$, $\sigma_2 = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto \mathbf{s}_n(t + s, y) \in \mathbb{R}^{d \times m})$, $\mathbb{F} = (\mathbb{F}_{t+s})_{s \in [0, T-t]}$, $W = ([0, T - t] \times \Omega \ni (s, \omega) \mapsto W_{t+s}(\omega) - W_t(\omega) \in \mathbb{R}^m)$, $X^{(1)} = ([0, T - t] \times \Omega \ni (s, \omega) \mapsto X_{t+s}^{t,x}(\omega) \in \mathcal{O})$, $X^{(2)} = ([0, T - t] \times \Omega \ni (s, \omega) \mapsto \mathfrak{X}_{t+s}^{n,t,x}(\omega) \in \mathcal{O})$, $\tau = \tau^{n,t,x} - t$ for $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ in the notation of Lemma 3.5) ensure for all $n \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ that

$$\mathbb{P}(\forall s \in [t, T] : \mathbb{1}_{\{s \leq \tau^{n,t,x}\}} \|\mathfrak{X}_s^{n,t,x} - X_s^{t,x}\| = 0) = 1. \quad (142)$$

This, Markov's inequality, and Lemma 3.1 (with $T = T - t$, $\mu = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto \mu(t + s, y) \in \mathbb{R}^d)$, $\sigma = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto \sigma(t + s, y) \in \mathbb{R}^{d \times m})$, $V = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto V(t + s, y) \in [0, \infty))$, $\mathbb{F} = (\mathbb{F}_{t+s})_{s \in [0, T-t]}$, $W = ([0, T - t] \times \Omega \ni (s, \omega) \mapsto W_{t+s}(\omega) - W_t(\omega) \in \mathbb{R}^m)$, $X = ([0, T - t] \times \Omega \ni (s, \omega) \mapsto X_{t+s}^{t,x}(\omega) \in \mathcal{O})$, $\tau = \tau^{k,t,x} - t$ for $k \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$ in the notation of Lemma 3.1) imply for all $\varepsilon \in (0, \infty)$, $k \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathcal{O}$, $s \in [t, T]$ that

$$\begin{aligned} & \mathbb{P}(\|\mathfrak{X}_s^{k,t,x} - X_s^{t,x}\| \geq \varepsilon) \\ & \leq \mathbb{P}(\tau^{k,t,x} < s) \leq \mathbb{P}(V(\tau^{k,t,x}, X_{\tau^{k,t,x}}^{t,x}) \geq k) \leq \frac{1}{k} \mathbb{E}[V(\tau^{k,t,x}, X_{\tau^{k,t,x}}^{t,x})] \leq \frac{1}{k} V(t, x). \end{aligned} \quad (143)$$

Furthermore, observe that Lemma 3.6 ensures that there exist real numbers $c_k \in [0, \infty)$, $k \in \mathbb{N}$, which satisfy that for every $k, n \in \mathbb{N}$, $s \in [\mathbf{t}_0, T]$ it holds that

$$\mathbb{E} \left[\left\| \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} \right\|^2 \right] \leq c_k [|\mathbf{t}_n - \mathbf{t}_0| + \|\mathfrak{r}_n - \mathfrak{r}_0\|^2]. \quad (144)$$

Moreover, observe that (141) ensures that for all $k, n \in \mathbb{N}$, $s \in [\mathbf{t}_0, T]$ it holds \mathbb{P} -a.s. that

$$\begin{aligned} \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} - \mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} &= \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} - \mathfrak{X}_s^{k, \mathbf{t}_0, \mathfrak{r}_0} \\ &= \int_s^{\max\{s, \mathbf{t}_n\}} \mathbf{m}_k(r, \mathfrak{X}_r^{k, \mathbf{t}_0, \mathfrak{r}_0}) dr + \int_s^{\max\{s, \mathbf{t}_n\}} \mathfrak{s}_k(r, \mathfrak{X}_r^{k, \mathbf{t}_0, \mathfrak{r}_0}) dW_r. \end{aligned} \quad (145)$$

Minkowski's inequality, the fact that $\mathbf{m}_k : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $k \in \mathbb{N}$, and $\mathfrak{s}_k : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$, $k \in \mathbb{N}$, are compactly supported continuous functions, and Itô's isometry hence imply that for all $k, n \in \mathbb{N}$, $s \in [\mathbf{t}_0, T]$ it holds that

$$\begin{aligned} &\left(\mathbb{E} \left[\left\| \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} - \mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} \right\|^2 \right] \right)^{1/2} \\ &\leq \int_s^{\max\{s, \mathbf{t}_n\}} \left(\mathbb{E} \left[\|\mathbf{m}_k(r, \mathfrak{X}_r^{k, \mathbf{t}_0, \mathfrak{r}_0})\|^2 \right] \right)^{1/2} dr + \left(\int_s^{\max\{s, \mathbf{t}_n\}} \mathbb{E} \left[\|\mathfrak{s}_k(r, \mathfrak{X}_r^{k, \mathbf{t}_0, \mathfrak{r}_0})\|^2 \right] dr \right)^{1/2} \\ &\leq |\max\{0, \mathbf{t}_n - s\}|^{1/2} \left[\sqrt{T} \left(\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \|\mathbf{m}_k(t, x)\| \right) + \left(\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \|\mathfrak{s}_k(t, x)\| \right) \right]. \end{aligned} \quad (146)$$

This, the fact that for all $a, b \in \mathbb{R}$ it holds that $(a + b)^2 \leq 2a^2 + 2b^2$, and (144) ensure that there exist real numbers $\mathbf{c}_k \in [0, \infty)$, $k \in \mathbb{N}$, which satisfy for every $k, n \in \mathbb{N}$, $s \in [\mathbf{t}_0, T]$ that

$$\begin{aligned} \mathbb{E} \left[\left\| \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} \right\|^2 \right] &\leq 2\mathbb{E} \left[\left\| \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} \right\|^2 \right] + 2\mathbb{E} \left[\left\| \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} - \mathfrak{X}_s^{k, \mathbf{t}_0, \mathfrak{r}_0} \right\|^2 \right] \\ &\leq \mathbf{c}_k [|\mathbf{t}_n - \mathbf{t}_0| + \|\mathfrak{r}_n - \mathfrak{r}_0\|^2 + \max\{0, \mathbf{t}_n - s\}]. \end{aligned} \quad (147)$$

In addition, observe that (141) ensures that for all $k, n \in \mathbb{N}$, $s \in [0, \mathbf{t}_0]$ it holds \mathbb{P} -a.s. that

$$\begin{aligned} \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} &= \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{r}_0 \\ &= \mathfrak{r}_n - \mathfrak{r}_0 + \int_{\mathbf{t}_n}^{\max\{s, \mathbf{t}_n\}} \mathbf{m}_k(r, \mathfrak{X}_r^{k, \mathbf{t}_n, \mathfrak{r}_n}) dr + \int_{\mathbf{t}_n}^{\max\{s, \mathbf{t}_n\}} \mathfrak{s}_k(r, \mathfrak{X}_r^{k, \mathbf{t}_n, \mathfrak{r}_n}) dW_r. \end{aligned} \quad (148)$$

Minkowski's inequality, the fact that $\mathbf{m}_k : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $k \in \mathbb{N}$, and $\mathfrak{s}_k : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$, $k \in \mathbb{N}$, are compactly supported continuous functions, and Itô's isometry hence imply that for all $k, n \in \mathbb{N}$, $s \in [0, \mathbf{t}_0]$ it holds that

$$\begin{aligned} &\left(\mathbb{E} \left[\left\| \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} \right\|^2 \right] \right)^{1/2} \leq \|\mathfrak{r}_n - \mathfrak{r}_0\| \\ &+ \int_{\mathbf{t}_n}^{\max\{s, \mathbf{t}_n\}} \left(\mathbb{E} \left[\|\mathbf{m}_k(r, \mathfrak{X}_r^{k, \mathbf{t}_n, \mathfrak{r}_n})\|^2 \right] \right)^{1/2} dr + \left(\int_{\mathbf{t}_n}^{\max\{s, \mathbf{t}_n\}} \mathbb{E} \left[\|\mathfrak{s}_k(r, \mathfrak{X}_r^{k, \mathbf{t}_n, \mathfrak{r}_n})\|^2 \right] dr \right)^{1/2} \\ &\leq \|\mathfrak{r}_n - \mathfrak{r}_0\| + |\max\{0, s - \mathbf{t}_n\}|^{1/2} \left[\sqrt{T} \left(\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \|\mathbf{m}_k(t, x)\| \right) + \left(\sup_{t \in [0, T]} \sup_{x \in \mathcal{O}} \|\mathfrak{s}_k(t, x)\| \right) \right]. \end{aligned} \quad (149)$$

This and (147) ensure that there exist real numbers $\mathbf{c}_k \in [0, \infty)$, $k \in \mathbb{N}$, which satisfy for every $k, n \in \mathbb{N}$, $s \in [0, T]$ that

$$\begin{aligned} & \mathbb{E} \left[\left\| \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} \right\|^2 \right] \\ & \leq \mathbf{c}_k \left[|\mathbf{t}_n - \mathbf{t}_0| + \|\mathfrak{r}_n - \mathfrak{r}_0\|^2 + \mathbb{1}_{[0, \mathbf{t}_0]}(s) \max\{0, s - \mathbf{t}_n\} + \mathbb{1}_{[\mathbf{t}_0, T]}(s) \max\{0, \mathbf{t}_n - s\} \right]. \end{aligned} \quad (150)$$

Combining this with Markov's inequality and (143) demonstrates that for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ it holds that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left[\mathbb{P} \left(\|X_{\max\{s, \mathbf{t}_n\}}^{\mathbf{t}_n, \mathfrak{r}_n} - X_{\max\{s, \mathbf{t}_0\}}^{\mathbf{t}_0, \mathfrak{r}_0}\| \geq \varepsilon \right) \right] \\ & \leq \inf_{k \in \mathbb{N}} \left(\limsup_{n \rightarrow \infty} \left[\mathbb{P} \left(\|X_{\max\{s, \mathbf{t}_n\}}^{\mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n}\| \geq \frac{\varepsilon}{3} \right) \right. \right. \\ & \quad \left. \left. + \mathbb{P} \left(\|\mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0}\| \geq \frac{\varepsilon}{3} \right) + \mathbb{P} \left(\|\mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} - X_{\max\{s, \mathbf{t}_0\}}^{\mathbf{t}_0, \mathfrak{r}_0}\| \geq \frac{\varepsilon}{3} \right) \right] \right) \\ & \leq \inf_{k \in \mathbb{N}} \left(\limsup_{n \rightarrow \infty} \left[\frac{V(\mathbf{t}_n, \mathfrak{r}_n)}{k} + \frac{9}{\varepsilon^2} \mathbb{E} \left[\left\| \mathfrak{X}_{\max\{s, \mathbf{t}_n\}}^{k, \mathbf{t}_n, \mathfrak{r}_n} - \mathfrak{X}_{\max\{s, \mathbf{t}_0\}}^{k, \mathbf{t}_0, \mathfrak{r}_0} \right\|^2 \right] + \frac{V(\mathbf{t}_0, \mathfrak{r}_0)}{k} \right] \right) \\ & \leq \inf_{k \in \mathbb{N}} \left(\limsup_{n \rightarrow \infty} \left[\frac{V(\mathbf{t}_n, \mathfrak{r}_n)}{k} + \frac{9\mathbf{c}_k}{\varepsilon^2} \left(|\mathbf{t}_n - \mathbf{t}_0| + \|\mathfrak{r}_n - \mathfrak{r}_0\|^2 \right. \right. \right. \\ & \quad \left. \left. \left. + \mathbb{1}_{[0, \mathbf{t}_0]}(s) \max\{0, s - \mathbf{t}_n\} + \mathbb{1}_{[\mathbf{t}_0, T]}(s) \max\{0, \mathbf{t}_n - s\} \right) + \frac{V(\mathbf{t}_0, \mathfrak{r}_0)}{k} \right] \right) \\ & = \inf_{k \in \mathbb{N}} \left(\frac{2V(\mathbf{t}_0, \mathfrak{r}_0)}{k} \right) = 0. \end{aligned} \quad (151)$$

This demonstrates (132). The proof of Lemma 3.7 is thus completed. \square

The next result, Theorem 3.8 below, is the main result of this article. It is an application of Theorem 2.9. Lemmas 3.1 and 3.7 above ensure that the crucial hypotheses of Theorem 2.9 are satisfied in the setting of Theorem 3.8.

Theorem 3.8. *Let $d, m \in \mathbb{N}$, $L, T \in (0, \infty)$, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the standard scalar product on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d \times m}$, let $\mathcal{O} \subseteq \mathbb{R}^d$ be a non-empty open set, for every $r \in (0, \infty)$ let $O_r \subseteq \mathcal{O}$ satisfy $O_r = \{x \in \mathcal{O}: \|x\| \leq r \text{ and } \{y \in \mathbb{R}^d: \|y - x\| < 1/r\} \subseteq \mathcal{O}\}$, let $\mu \in C([0, T] \times \mathcal{O}, \mathbb{R}^d)$, $\sigma \in C([0, T] \times \mathcal{O}, \mathbb{R}^{d \times m})$ satisfy for all $r \in (0, \infty)$ that*

$$\sup \left(\left\{ \frac{\|\mu(t, x) - \mu(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\|}{\|x - y\|}: t \in [0, T], x, y \in O_r, x \neq y \right\} \cup \{0\} \right) < \infty, \quad (152)$$

let $f \in C([0, T] \times \mathcal{O} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathcal{O}, \mathbb{R})$, $V \in C^{1,2}([0, T] \times \mathcal{O}, (0, \infty))$, assume for all $t \in [0, T]$, $x \in \mathcal{O}$, $v, w \in \mathbb{R}$ that $|f(t, x, v) - f(t, x, w)| \leq L|v - w|$ and

$$\left(\frac{\partial V}{\partial t} \right)(t, x) + \frac{1}{2} \text{Trace}(\sigma(t, x)[\sigma(t, x)]^* (\text{Hess}_x V)(t, x)) + \langle \mu(t, x), (\nabla_x V)(t, x) \rangle \leq 0, \quad (153)$$

assume that $\sup_{r \in (0, \infty)} [\inf_{t \in [0, T]} \inf_{x \in \mathcal{O} \setminus O_r} V(t, x)] = \infty$ and $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathcal{O} \setminus O_r} \left(\frac{|f(t, x, 0)|}{V(t, x)} + \frac{|g(x)|}{V(t, x)} \right)] = 0$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, and for every $t \in [0, T]$,

$x \in \mathcal{O}$ let $X^{t,x} = (X_s^{t,x})_{s \in [t,T]}: [t,T] \times \Omega \rightarrow \mathcal{O}$ be an $(\mathbb{F}_s)_{s \in [t,T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $s \in [t,T]$ it holds \mathbb{P} -a.s. that

$$X_s^{t,x} = x + \int_t^s \mu(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r. \quad (154)$$

Then there exists a unique $u \in C([0,T] \times \mathcal{O}, \mathbb{R})$ such that

(i) it holds that

$$\limsup_{r \rightarrow \infty} \left[\sup_{t \in [0,T]} \sup_{x \in \mathcal{O} \setminus \mathcal{O}_r} \left(\frac{|u(t,x)|}{V(t,x)} \right) \right] = 0 \quad (155)$$

and

(ii) it holds for all $t \in [0,T]$, $x \in \mathcal{O}$ that

$$u(t,x) = \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds \right]. \quad (156)$$

Proof of Theorem 3.8. First, note that Lemma 3.1 (with $T = T - t$, $\mu = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto \mu(t + s, y) \in \mathbb{R}^d$), $\sigma = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto \sigma(t + s, y) \in \mathbb{R}^{d \times m}$), $V = ([0, T - t] \times \mathcal{O} \ni (s, y) \mapsto V(t + s, y) \in [0, \infty)$), $\mathbb{F} = (\mathbb{F}_{t+s})_{s \in [0, T-t]}$, $W = ([0, T - t] \times \Omega \ni (s, \omega) \mapsto W_{t+s}(\omega) - W_t(\omega) \in \mathbb{R}^m$), $X = ([0, T - t] \times \Omega \ni (s, \omega) \mapsto X_{t+s}^{t,x}(\omega) \in \mathcal{O})$ for $t \in [0, T]$, $x \in \mathcal{O}$ in the notation of Lemma 3.1) ensures that for all $t \in [0, T]$, $s \in [t, T]$, $x \in \mathcal{O}$ it holds that

$$\mathbb{E}[V(s, X_s^{t,x})] \leq V(t, x). \quad (157)$$

Next observe that Lemma 3.7 ensures that for all $\varepsilon \in (0, \infty)$, $s \in [0, T]$ and all $(t_n, x_n) \in [0, T] \times \mathcal{O}$, $n \in \mathbb{N}_0$, with $\limsup_{n \rightarrow \infty} [|t_n - t_0| + \|x_n - x_0\|] = 0$ it holds that $\limsup_{n \rightarrow \infty} [\mathbb{P}(\|X_{\max\{s, t_n\}}^{t_n, x_n} - X_{\max\{s, t_0\}}^{t_0, x_0}\| \geq \varepsilon)] = 0$. Combining this with (157) and Theorem 2.9 demonstrates that there exists a unique $u \in C([0, T] \times \mathcal{O}, \mathbb{R})$ which satisfies that for all $t \in [0, T]$, $x \in \mathcal{O}$ it holds that $\limsup_{r \rightarrow \infty} [\sup_{s \in [0, T]} \sup_{y \in \mathcal{O} \setminus \mathcal{O}_r} (\frac{|u(s,y)|}{V(s,y)})] = 0$ and

$$u(t,x) = \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds \right]. \quad (158)$$

This establishes Items (i) and (ii). The proof of Theorem 3.8 is thus completed. \square

Lemma 3.2 implies the following corollary of Theorem 3.8 in the situation in which the drift and diffusion coefficients $\mu: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^d$ and $\sigma: [0, T] \times \mathcal{O} \rightarrow \mathbb{R}^{d \times m}$ depend only on the spatial variable $x \in \mathcal{O}$ and are independent of the time variable $t \in [0, T]$. For the sake of simplicity we take the spatial domain \mathcal{O} to be \mathbb{R}^d in Corollary 3.9 below.

Corollary 3.9. *Let $d, m \in \mathbb{N}$, $L, T \in (0, \infty)$, $\rho \in \mathbb{R}$, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the standard scalar product on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ be locally Lipschitz continuous, let $f \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}^d, \mathbb{R})$, $V \in C^2(\mathbb{R}^d, (0, \infty))$, assume for all $t \in [0, T]$, $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(t, x, v) - f(t, x, w)| \leq L|v - w|$ and*

$$\frac{1}{2} \text{Trace}(\sigma(x)[\sigma(x)]^*(\text{Hess } V)(x)) + \langle \mu(x), (\nabla V)(x) \rangle \leq \rho V(x), \quad (159)$$

assume that $\sup_{r \in (0, \infty)} [\inf_{x \in \mathbb{R}^d, \|x\| > r} V(x)] = \infty$ and $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d, \|x\| > r} (\frac{|f(t,x,0)| + |g(x)|}{V(x)})] = 0$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let

$W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, and for every $t \in [0, T]$, $x \in \mathbb{R}^d$ let $X^{t,x} = (X_s^{t,x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathbb{R}^d$ be an $(\mathbb{F}_s)_{s \in [t, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $s \in [t, T]$ it holds \mathbb{P} -a.s. that

$$X_s^{t,x} = x + \int_t^s \mu(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r. \quad (160)$$

Then there exists a unique $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ such that

(i) it holds that

$$\limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{\substack{x \in \mathbb{R}^d \\ \|x\| > r}} \left(\frac{|u(t, x)|}{V(x)} \right) \right] = 0 \quad (161)$$

and

(ii) for all $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that

$$u(t, x) = \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds \right]. \quad (162)$$

Proof of Corollary 3.9. Throughout this proof let $\mathbb{V}: [0, T] \times \mathbb{R}^d \rightarrow (0, \infty)$ satisfy for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that $\mathbb{V}(t, x) = e^{-\rho t} V(x)$. Observe that Lemma 3.2 (with $\mathcal{O} = \mathbb{R}^d$, $\mu = ([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \mu(x) \in \mathbb{R}^d)$, $\sigma = ([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \sigma(x) \in \mathbb{R}^{d \times m})$ in the notation of Lemma 3.2) ensures that for all $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that $\mathbb{V} \in C^{1,2}([0, T] \times \mathbb{R}^d, (0, \infty))$ and

$$\left(\frac{\partial \mathbb{V}}{\partial t} \right)(t, x) + \frac{1}{2} \text{Trace}(\sigma(x)[\sigma(x)]^* (\text{Hess}_x \mathbb{V})(t, x)) + \langle \mu(x), (\nabla_x \mathbb{V})(t, x) \rangle \leq 0. \quad (163)$$

Next, observe that the hypothesis that $\sup_{r \in (0, \infty)} [\inf_{x \in \mathbb{R}^d, \|x\| > r} V(x)] = \infty$ implies that

$$\sup_{r \in (0, \infty)} \left[\inf_{t \in [0, T]} \inf_{x \in \mathbb{R}^d, \|x\| > r} \mathbb{V}(t, x) \right] = \infty. \quad (164)$$

Furthermore, observe that the hypothesis that $\inf_{r \in (0, \infty)} [\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d, \|x\| > r} \left(\frac{|f(t, x, 0)| + |g(x)|}{V(x)} \right)] = 0$ demonstrates that

$$\inf_{r \in (0, \infty)} \left[\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d, \|x\| > r} \left(\frac{|f(t, x, 0)|}{\mathbb{V}(t, x)} + \frac{|g(x)|}{\mathbb{V}(T, x)} \right) \right] = 0. \quad (165)$$

Theorem 3.8 (with $\mathcal{O} = \mathbb{R}^d$, $\mu = ([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \mu(x) \in \mathbb{R}^d)$, $\sigma = ([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \sigma(x) \in \mathbb{R}^{d \times m})$, $V = \mathbb{V}$ in the notation of Theorem 3.8) and (164) hence ensure that there exists a unique $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ which satisfies that

(I) it holds that

$$\limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{\substack{x \in \mathbb{R}^d \\ \|x\| > r}} \left(\frac{|u(t, x)|}{\mathbb{V}(t, x)} \right) \right] = 0 \quad (166)$$

and

(II) it holds for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$u(t, x) = \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, u(s, X_s^{t,x})) ds \right]. \quad (167)$$

Next note that Item (I) implies that

$$\limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{\substack{x \in \mathbb{R}^d \\ \|x\| > r}} \left(\frac{|u(t, x)|}{V(x)} \right) \right] = 0. \quad (168)$$

This establishes Item (i). Moreover, note that Item (II) establishes Item (ii). The proof of Corollary 3.9 is thus completed. \square

Finally, in Corollary 3.10 below, we specialize in the setting of Corollary 3.9 to the situation in which the drift and diffusion coefficients $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ satisfy a coercivity condition and the nonlinearity $f: [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ as well as the terminal condition $g: \mathbb{R}^d \rightarrow \mathbb{R}$ are at most polynomially growing with respect to the spatial variable $x \in \mathbb{R}^d$. Suitable choices for the Lyapunov-type function V are provided by Lemma 3.3.

Corollary 3.10 (Existence and uniqueness of at most polynomially growing solutions of SFPEs). *Let $d, m \in \mathbb{N}$, $L, T \in (0, \infty)$, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the standard scalar product on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$ be the standard norm on \mathbb{R}^d , let $\|\cdot\|: \mathbb{R}^{d \times m} \rightarrow [0, \infty)$ be the Frobenius norm on $\mathbb{R}^{d \times m}$, let $\mu: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ be locally Lipschitz continuous, let $f \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}^d, \mathbb{R})$ be at most polynomially growing, assume for all $t \in [0, T]$, $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $\max\{\langle x, \mu(x) \rangle, \|\sigma(x)\|^2\} \leq L(1 + \|x\|^2)$ and $|f(t, x, v) - f(t, x, w)| \leq L|v - w|$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $W: [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion, and for every $t \in [0, T]$, $x \in \mathbb{R}^d$ let $X^{t, x} = (X_s^{t, x})_{s \in [t, T]}: [t, T] \times \Omega \rightarrow \mathbb{R}^d$ be an $(\mathbb{F}_s)_{s \in [t, T]}$ -adapted stochastic process with continuous sample paths which satisfies that for all $s \in [t, T]$ it holds \mathbb{P} -a.s. that*

$$X_s^{t, x} = x + \int_t^s \mu(X_r^{t, x}) dr + \int_t^s \sigma(X_r^{t, x}) dW_r. \quad (169)$$

Then there exists a unique $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ such that

- (i) it holds that u is at most polynomially growing and
- (ii) it holds for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$u(t, x) = \mathbb{E} \left[g(X_T^{t, x}) + \int_t^T f(s, X_s^{t, x}, u(s, X_s^{t, x})) ds \right]. \quad (170)$$

Proof of Corollary 3.10. Throughout this proof let $\rho_q \in (0, \infty)$, $q \in (0, \infty)$, satisfy for every $q \in (0, \infty)$ that $\rho_q = \frac{qL}{2} \max\{q + 1, 3\}$, let $p \in (0, \infty)$ satisfy that $\sup_{t \in [0, T]} \sup_{y \in \mathbb{R}^d} \left[\frac{|f(t, y, 0)| + |g(y)|}{1 + \|y\|^p} \right] < \infty$, and let $V_q: \mathbb{R}^d \rightarrow \mathbb{R}$, $q \in (0, \infty)$, satisfy for all $q \in (0, \infty)$, $x \in \mathbb{R}^d$ that $V_q(x) = [1 + \|x\|^2]^{q/2}$. Note that the fact that $\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \left[\frac{|f(t, x, 0)| + |g(x)|}{1 + \|x\|^p} \right] < \infty$ implies that for all $q \in (p, \infty)$ it holds that

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{\substack{x \in \mathbb{R}^d \\ \|x\| > r}} \left(\frac{|f(t, x, 0)| + |g(x)|}{V_q(x)} \right) \right] \\ &= \limsup_{r \rightarrow \infty} \left[\sup_{t \in [0, T]} \sup_{\substack{x \in \mathbb{R}^d \\ \|x\| > r}} \left(\left[\frac{|f(t, x, 0)| + |g(x)|}{1 + \|x\|^p} \right] \left[\frac{1 + \|x\|^p}{V_q(x)} \right] \right) \right] \\ &\leq \left[\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} \left(\frac{|f(t, x, 0)| + |g(x)|}{1 + \|x\|^p} \right) \right] \left[\limsup_{r \rightarrow \infty} \left(\sup_{t \in [0, T]} \sup_{\substack{x \in \mathbb{R}^d \\ \|x\| > r}} \left(\frac{1 + \|x\|^p}{V_q(x)} \right) \right) \right] = 0. \end{aligned} \quad (171)$$

Moreover, observe that for all $q \in (p, \infty)$ it holds that

$$\sup_{r \in (0, \infty)} \left[\inf_{\substack{x \in \mathbb{R}^d, \\ \|x\| > r}} V_q(x) \right] = \infty. \quad (172)$$

Next note that Lemma 3.3 ensures for all $q \in (0, \infty)$, $x \in \mathbb{R}^d$ that

$$\frac{1}{2} \text{Trace}(\sigma(x)[\sigma(x)]^* (\text{Hess } V_q)(x)) + \langle \mu(x), (\nabla V_q)(x) \rangle \leq \rho_q V_q(x). \quad (173)$$

Combining this with (171), (172), and Corollary 3.9 (with $V = V_{2p}$ in the notation of Corollary 3.9) yields that there exists a unique $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ which satisfies for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that $\limsup_{r \rightarrow \infty} [\sup_{s \in [0, T]} \sup_{y \in \mathbb{R}^d, \|y\| > r} (\frac{|u(s, y)|}{V_{2p}(y)})] = 0$ and $u(t, x) = \mathbb{E}[g(X_T^{t, x}) + \int_t^T f(s, X_s^{t, x}, u(s, X_s^{t, x})) ds]$. In particular, this ensures that $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is at most polynomially growing. This establishes that $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ satisfies Items (i) and (ii). It remains to prove that $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the only continuous function which satisfies Items (i) and (ii). For this, let $v \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ be an at most polynomially growing function which satisfies for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that $v(t, x) = \mathbb{E}[g(X_T^{t, x}) + \int_t^T f(s, X_s^{t, x}, v(s, X_s^{t, x})) ds]$. The fact that $v: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is at most polynomially growing ensures that there exists $q \in (0, \infty)$ which satisfies that $\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}^d} [\frac{|v(t, x)|}{1 + \|x\|^q}] < \infty$. This implies that $u, v \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$ satisfy for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that $\limsup_{r \rightarrow \infty} [\sup_{s \in [0, T]} \sup_{y \in \mathbb{R}^d, \|y\| > r} (\frac{|u(s, y)| + |v(s, y)|}{V_{\max\{2q, 2p\}}(y)})] = 0$, $u(t, x) = \mathbb{E}[g(X_T^{t, x}) + \int_t^T f(s, X_s^{t, x}, u(s, X_s^{t, x})) ds]$, and

$$v(t, x) = \mathbb{E} \left[g(X_T^{t, x}) + \int_t^T f(s, X_s^{t, x}, v(s, X_s^{t, x})) ds \right]. \quad (174)$$

Corollary 3.9 (with $V = V_{\max\{2q, 2p\}}$ in the notation of Corollary 3.9) hence ensures that $u = v$. This establishes that $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the unique continuous function which satisfies Items (i) and (ii). The proof of Corollary 3.10 is thus completed. \square

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