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Moment Inequalities for Sums of Dependent Random Variables under Projective Conditions

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Abstract We obtain precise constants in the Marcinkiewicz-Zygmund inequality for martingales in $\mathbb{L}^p$ for $p > 2$ and a new Rosenthal type inequality for stationary martingale differences for p in $]2, 3]$. The Rosenthal inequality is then extended to stationary and adapted sequences. As in Peligrad et al. (Proc. Am. Math. Soc. 135:541–550, 2007), the bounds are expressed in terms of $\mathbb{L}^p$-norms of conditional expectations with respect to an increasing field of sigma algebras. Some applications to a particular Markov chain are given.

Keywords Martingale · Moment inequality · Stationary sequences · Projective criteria · Rosenthal inequality

Mathematics Subject Classification (2000) 60 F 05 · 60 F 17

1 Introduction

In this paper we give new moment inequalities for partial sums of dependent sequences. For $p \geq 1$, let $\mathbb{L}^p$ be the space of real-valued random variables with finite absolute moment of order p. $\|Z\|_p$ denotes the $\mathbb{L}^p$-norm of Z. Let $p > 2$ and $(X_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued random variables in $\mathbb{L}^p$. Set $S_n = X_1 + X_2 + \cdots + X_n$. Our aim is to provide the best possible bounds on $\|S_n\|_p$. under various dependence assumptions.

The first part of the paper will be devoted to martingale difference sequences. These sequences play an important role in establishing moment inequalities for partial sums of stationary sequences, as shown by Peligrad, Utev and Wu [13]. Let us

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recall the classical results on this subject. Burkholder [3] proved that, for any $p > 1$,

$$\|S_n\|_p \le C_p \|(X_1^2 + X_2^2 + \cdots + X_n^2)^{1/2}\|_p. \tag{1.1}$$

In his paper in Astérisque, Burkholder [4] obtains (1.1) with $C_p = p - 1$ for $p > 2$ and $C_p = 1/(p-1)$ for $p < 2$. He also proves that this constant is optimal for $p > 2$. From (1.1), for any $p > 2$ the following Marcinkiewicz-Zygmund type inequality holds:

$$\|S_n\|_p^2 \le c_p\big(\|X_1\|_p^2 + \|X_2\|_p^2 + \cdots + \|X_n\|_p^2\big), \tag{1.2}$$

with $c_p = (p-1)^2$. Recently, Nagaev [10] obtains (1.2) with the improved constant $c_p = p(p-1)/2$. However the dependence in p is suboptimal as p tends to ∞ as shown in [15] in the stationary case and in [16]. In Sect. 2 we prove that (1.2) holds with $c_p = p - 1$ and that this constant cannot be improved. We also give an application of this inequality to deviation inequalities for sequences of martingale differences with subgaussian tails.

For $p > 2$, Peligrad, Utev and Wu [13] generalize the Marcinkiewicz-Zygmund inequality to stationary weakly dependent sequences. In order to state their inequality, we need further notation and definitions. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $T : \Omega \to \Omega$ be a bijective bi-measurable transformation preserving $\mathbb{P}$. Let $\mathcal{F}_0$ be a σ-algebra of $\mathcal{A}$ satisfying $\mathcal{F}_0 \subset T^{-1}(\mathcal{F}_0)$ and define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let X be an integrable and centered $\mathcal{F}_0$-measurable real-valued random variable. Define the sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X \circ T^i$. Then $(X_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$. From now on we denote by $E_i(Z)$ the conditional expectation of Z with respect to $\mathcal{F}_i$.

From Proposition 2.1, in [12, p. 800], for stationary sequences,

$$\|S_n\|_2 \le \sigma_N \sqrt{n} \quad \text{for any } \ n \le 2^N, \ \text{with } \sigma_N = \|X_0\|_2 + \frac{1}{2}\sum_{L=0}^{N-1} 2^{-L/2} \|E_0(S_{2^L})\|_2. \tag{1.3}$$

Peligrad, Utev and Wu [13] extend this bound has been to the case $p > 2$. Let $S_n^* = \max(|S_1|, |S_2|, \ldots, |S_n|)$: they prove that

$$\|S_n^*\|_p \le C\sqrt{pn}\left(\|X_1\|_p + \sum_{L=0}^{N-1} 2^{-L/2}\|E_0(S_{2^L})\|_p\right) \quad \text{for any } n \le 2^N, \tag{1.4}$$

for some explicit universal constant C. The case $p < 2$ is studied in [18], who prove that

$$\|S_{2^N}^*\|_p \le n^{1/p}\left(B_p\|X_1\|_p + B_p' \sum_{L=0}^{N-1} 2^{-L/p}\|E_0(S_{2^L})\|_p\right), \tag{1.5}$$

with $B_p = 18p^{5/2}(p-1)^{-3/2}$ and $B_p' = 2^{-1/p} + B_p(1 + 2^{(p-1)/p}$. Although the constant B_p appearing in (1.5) may be improved, the optimal constant tends to ∞ as p decreases to 1, due to the fact that the proof needs a maximal version of the Marcinkiewicz-Zygmund inequality.

Inequality (1.4) can be applied to get exponential inequalities for partial sums of weakly dependent sequences, as in [13]. Nevertheless the variance of S_n does not appear in the right hand side of the inequality. This is the reason why we will try to get Rosenthal type inequalities for martingale difference sequences. For independent random variables in $\mathbb{L}^p$, Rosenthal's inequality has the following formulation: there exists positive constants a_p and b_p such that

$$\mathbb{E}(|S_n|^p) \le a_p \|S_n\|_2^p + b_p \sum_{i=1}^{n} \mathbb{E}(|X_i|^p).$$

For p in $]2,4]$ and independent symmetric random variables, Rosenthal's inequality holds with $b_p = 1$ and $a_p = \mathbb{E}(|Y|^p)$, where Y has the standard normal distribution (confer Figiel et al. [8] for more about the constants). These inequalities have applications in nonparametric statistics, as proved in [2]. Some extensions of Rosenthal's inequality to absolutely regular sequences with applications to nonparametric statistics may be found in [16]. For strongly mixing sequences, we refer to [6] and [15] for more about moment inequalities for partial sums.

For p in $]2,4]$ and stationary sequences $(X_i)_{i\in\mathbb{Z}}$ of martingale differences in $\mathbb{L}^p$ with $\operatorname{Var} X_0 = \sigma^2$, (1.4) may be used to prove a Rosenthal type inequality in the following way. Starting from (1.1) and applying (1.5) with the exponent $p/2$,

$$\begin{aligned}\|S_n\|_p^2 &\le (p-1)^2\big(n\sigma^2 + \|X_1^2 + X_2^2 + \cdots + X_n^2 - n\sigma^2\|_{p/2}\big)\\ &\le (p-1)^2\big(n\sigma^2 + n^{2/p}(B_{p/2}\|X_1^2 - \sigma^2\|_{p/2} + B'_{p/2}\Delta_N)\big),\end{aligned} \tag{1.6}$$

for any $n \le 2^N$, where

$$\Delta_N = \sum_{L=0}^{N-1} 2^{-2L/p} \| E_0(S_{2^L}^2) - \mathbb{E}(S_{2^L}^2) \|_{p/2}. \tag{1.7}$$

The constants appearing in (1.6) are far from being optimal. Moreover it seems difficult to extend the method of [13] to get a Rosenthal type inequality for general stationary sequences. This is the reason why we will use a direct approach to get Rosenthal type inequalities. In order to avoid some technical difficulties, we will restrict our attention to exponents p in $]2,3]$. In Sect. 3, we will prove that, for any positive integer N and any n in $[2^N, 2^{N+1}[$

$$\mathbb{E}(|S_n|^p) \le a_p n^{p/2}\sigma_N^p + b_p n \mathbb{E}(|X_0|^p) + c_p n \Delta_N^{p/2} + d_p n D_N^p \tag{1.8}$$

for some explicit constants a_p, b_p, c_p and d_p, where

$$D_N = \sum_{L=0}^{N-1} 2^{-L/p} \|E_0(S_{2^L})\|_p. \tag{1.9}$$

Inequality (1.8) is an extension of Rosenthal's inequality if the sequences (Δ_N) and (D_N) are convergent. In Sect. 4, we apply these inequalities to additive functionals of some Harris recurrent irreducible Markov chain and we compare the results with previous inequalities in this particular setting.

2 A Marcinkiewicz-Zygmund Type Inequality for Martingales

In this section, we prove inequality (1.2) with $c_p = p - 1$. The main step of the proof is Proposition 2.1 below. Throughout this section $\mathbb{E}(Y \mid X)$ denotes the conditional expectation of Y conditionally to X.

Proposition 2.1 *Let $p > 2$ and X and Y be random variables in $\mathbb{L}^p$ such that $\mathbb{E}(Y \mid X) = 0$ almost surely. Then $\|X + Y\|_p^2 \le \|X\|_p^2 + (p-1)\|Y\|_p^2$.*

Remark 2.1 The constant $p - 1$ cannot be improved, at least for products of independent random variables. Actually, let ε be a symmetric sign, independent of X, and $Y_\alpha = \alpha\varepsilon X$. Then $\mathbb{E}(Y_\alpha \mid X) = 0$ and

$$\lim_{\alpha \searrow 0} \frac{\|X + Y_\alpha\|_p^2 - \|X\|_p^2}{\|Y_\alpha\|_p^2} = p - 1.$$

We refer to [10, pp. 155–157] for more about moment inequalities for products of independent random variables and for a survey of previous results

Proof of Proposition 2.1 Obviously Proposition 2.1 holds true if a.s. $X = 0$ or $Y = 0$. Hence we may assume that $\|X\|_p > 0$ and $\|Y\|_p > 0$. Next, dividing the random variables by $\|X\|_p$ if necessary, we may assume that $\|X\|_p = 1$. Let φ be defined on $[0, \infty[$ by $\varphi(t) = \|X + tY\|_p^p$. We have to prove that

$$\varphi(t) \le (1 + (p-1)\|Y\|_p^2 t^2)^{p/2}.$$

By the Taylor integral formula at order two applied to φ,

$$|X + tY|_p^p = |X|^p + pt|X|^{p-2}XY + p(p-1)\int_0^t (t-s)Y^2|X + sY|^{p-2}ds.$$

Since $\mathbb{E}(Y \mid X) = 0$, we infer that

$$\varphi(t) = 1 + p(p-1)\int_0^t (t-s)\mathbb{E}(Y^2|X + sY|^{p-2})ds.$$

Now, by the Hölder inequality,

$$\mathbb{E}(Y^2|X + sY|^{p-2}) \le \|X + sY\|_p^{p-2}\|Y\|_p^2,$$

whence

$$\varphi(t) \le 1 + p(p-1)\|Y\|_p^2 \int_0^t (t-s)(\varphi(s))^{1-2/p}ds. \tag{2.1}$$

Let ψ be defined on $[0, \infty[$ by

$$\psi(t) = 1 + p(p-1)\|Y\|_p^2 \int_0^t (t-s)(\varphi(s))^{1-2/p}ds.$$

Then ψ is a positive, convex and increasing function with

$$\psi'(t) = p(p-1)\|Y\|_p^2 \int_0^t (\varphi(s))^{1-2/p} ds \quad \text{and} \quad \psi''(t) = p(p-1)\|Y\|_p^2 (\varphi(t))^{1-2/p}.$$

Now, by (2.1),

$$\psi''(t) \le p(p-1)\|Y\|_p^2 (\psi(t))^{1-2/p}.$$

Multiplying this inequality by the positive function ψ', we obtain

$$2\psi'(t)\psi''(t) \le 2p(p-1)\|Y\|_p^2 \psi'(t)(\psi(t))^{1-2/p}.$$

Since $\psi'(0) = 0$, integrating this inequality between 0 and x and taking the square root, we get that

$$\psi'(x) \le p\|Y\|_p \sqrt{(\psi(x))^{2-2/p} - 1}\,. \tag{2.2}$$

In order to get a tractable differential inequality, we will need the elementary lemma below.

Lemma 2.1 *Let* $p > 2$. *Then* $\sqrt{y^{2-2/p} - 1} \le y^{1-2/p}\sqrt{(p-1)(y^{2/p} - 1)}$ *for any* $y \ge 1$.

Proof of Lemma 2.1 We may assume $y > 1$. Let $\alpha = 2/p$. Lemma 2.1 holds true if

$$y^{2-\alpha} - 1 \le (p-1)y^{2-2\alpha}(y^\alpha - 1),$$

which is equivalent to $1 - y^{-(2-\alpha)} \le (p-1)(1 - y^{-\alpha})$. Now, from the convexity of the map $a \to 1 - y^{-a}$ for $y > 1$ and the fact that $2 - \alpha > \alpha$,

$$1 - y^{-(2-\alpha)} \le \alpha^{-1}(2-\alpha)(1 - y^{-\alpha}),$$

which completes the proof of Lemma 2.1, since $(2-\alpha)/\alpha = p - 1$. □

From (2.2) and Lemma 2.1 applied to $y = \psi(x)$,

$$\psi' \le p\sqrt{(p-1)}\psi^{1-2/p}\sqrt{\psi^{2/p} - 1}.$$

Hence, setting $z = \psi^{2/p}$, we get the differential inequality

$$z'(z-1)^{-1/2} \le 2\sqrt{(p-1)}\|Y\|_p. \tag{2.3}$$

Integrating this differential inequality between 0 and t and taking the squares, we get the upper bound $z(t) - 1 \le (p-1)\|Y\|_p^2 t^2$, whence

$$\psi(t) \le \left(1 + (p-1)\|Y\|_p^2 t^2\right)^{p/2}.$$

Proposition 2.1 follows then from (2.1), which tells that $\varphi(x) \le \psi(x)$. □

From Proposition 2.1, we get immediately the Marcinkiewicz-Zygmund type inequality below by induction on n.

Theorem 2.1 *Let $p > 2$ and $(S_n)_{n\geq 0}$ be a sequence of random variables in $\mathbb{L}^p$. Set $X_k = S_k - S_{k-1}$. Assume that $\mathbb{E}(X_k \mid S_{k-1}) = 0$ a.s. for any positive k. Then*

$$\|S_n\|_p^2 \leq \|S_0\|_p^2 + (p-1)\big(\|X_1\|_p^2 + \|X_2\|_p^2 + \cdots + \|X_n\|_p^2\big).$$

Remark 2.2 In the independent case, Whittle [17] obtained inequality (1.2) with $c_p = 8\pi^{-1/p}(\Gamma(\frac{p+1}{2}))^{2/p}$. Whittle's constant is equivalent to $4p/e$ as p tends to ∞. Nagaev [10] obtains (1.2) with $c_p = p(p-1)/2$, which improves on Whittle's constant for small values of p. Rio [15] obtained (1.2) in the stationary case with $2p$ instead of $p-1$. In the non stationary case, Ren and Liang [14] obtained the constant $18p$ in a slightly different form of inequality (1.1). Note that Theorem 2.1 may be used to improve the numerical constants in the inequalities of [13].

We now give an application of Theorem 2.1 to exponential inequalities. We emphasize that, for dependent sequences, the classical Hoeffding inequality may fail to hold for sums of random variables with subgaussian tails. We refer to [9] for more about lower bounds in the exponential inequalities for stationary sequences of martingale differences.

Corollary 2.2 *Let $(S_n)_{n\geq 0}$ be a sequence of random variables with $S_0 = 0$, adapted to some nondecreasing filtration $(\mathcal{F}_n)_{n\geq 0}$. Set $X_k = S_k - S_{k-1}$. Assume that $\mathbb{E}(X_k \mid \mathcal{F}_{k-1}) = 0$ and $\mathbb{E}(\exp(X_k^2)) \leq 1 + \zeta$ for any positive k, for some positive ζ. Let $S_n^* = \max(|S_1|, |S_2|, \ldots, |S_n|)$. Then, for any positive x,*

$$\mathbb{P}(S_n^* \geq (ne/2)^{1/2} x) \leq \zeta(2e)^{-1/2}(\cosh(x) - 1)^{-1}.$$

Proof Let $f(t) = \mathbb{E}(\cosh(tn^{-1/2}S_n)) - 1$. By Theorem 2.1, for any positive integer p,

$$n^{-p}\mathbb{E}(S_n^{2p}) \leq (2p-1)^p n^{-1} \sum_{i=1}^{n} \mathbb{E}(X_i^{2p}).$$

It follows that

$$f(t) \leq n^{-1} \sum_{i=1}^{n} \sum_{p=1}^{\infty} (1 - 1/(2p))^p (2p)^p \frac{p!}{(2p)!} \frac{t^{2p}}{p!} \mathbb{E}(X_i^{2p}).$$

Now, recall that the sequence $(n^{-1/2}(e/n)^n \exp(-1/(12n)) n!)_n$ is nondecreasing. Consequently

$$\frac{p!}{(2p)!} \leq 2^{-1/2} \exp(1/(24p)) (p/e)^p (2p/e)^{-2p}.$$

Hence

$$f(t) \le \frac{1}{n\sqrt{2}} \sum_{i=1}^{n} \sum_{p=1}^{\infty} \exp(p\log((2p-1)/2p) + 1/(24p)) \frac{(et^2)^p}{2^p p!} \mathbb{E}(X_i^{2p}).$$

Now $p\log((2p-1)/2p) \le -(1/2) - 1/(8p)$, whence

$$\begin{aligned} f(t) &\le \frac{1}{n\sqrt{2e}} \sum_{i=1}^{n} \sum_{p=1}^{\infty} \exp\left(-\frac{1}{12p}\right) \frac{(et^2)^p}{2^p p!} \mathbb{E}(X_i^{2p}) \\ &\le \frac{1}{n\sqrt{2e}} \sum_{i=1}^{n} \mathbb{E}\left(\exp\left(\frac{et^2 X_i^2}{2}\right) - 1\right). \end{aligned} \tag{2.4}$$

From (2.4) and the assumption that $\mathbb{E}(\exp(X_i^2) - 1) \le \zeta$, we get that

$$f((2/e)^{1/2}) \le \zeta(2e)^{-1/2}. \tag{2.5}$$

Corollary 2.2 follows then from both (2.5) and Doob's maximal inequality applied to the nonnegative submartingale $(\cosh((2/en)^{1/2} S_n) - 1)_n$. □

3 Rosenthal Type Inequalities under Projective Conditions

In this section, we will restrict our attention to upper bounds on $\|S_n\|_p$ for p in $]2, 3]$ and stationary sequences. The main result is Theorem 3.1 below.

Theorem 3.1 *Let p be any real in $]2, 3]$ and $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of real-valued random variables in $\mathbb{L}^p$. Then, for any positive integer N and any n in $[2^N, 2^{N+1}[$*

$$\mathbb{E}(|S_n|^p) \le \kappa \max(a_p n^{p/2} \sigma_N^p + 2n\mathbb{E}(|X_0|^p), n(c_p \Delta_N)^{p/2}, n(d_p D_N)^p),$$

where $\kappa = (1 - 2^{-1/p})^{-p}$ for $n \ne 2^N$, $\kappa = 1$ if $n = 2^N$, $a_p = 2(p-1)$, $c_p = p(p-1)$ and $d_p = 2p$.

Proof The main step is to bound up $\|S_n\|_p$ for $n = 2^N$. This will be done by induction on N. Set $S_0 = 0$. Clearly

$$|S_{2^N}|^p = \sum_{n\in[1,2^N]} (|S_n|^p - |S_{n-1}|^p). \tag{3.1}$$

We now introduce further notation and definitions.

Notation 3.1 Let ψ be defined by $\psi(x) = |x|^p$. Let S denote the sign function, which is defined by $S(x) = 1$ for $x > 0$, $S(0) = 0$ and $S(x) = -1$.

With these notations, $\psi'(x) = p|x|^{p-1}S(x)$ and $\psi''(x) = p(p-1)|x|^{p-2}$. By the Taylor integral formula at order two,

$$\psi(x+h) - \psi(x) = \psi'(x)h + \frac{1}{2}\psi''(x)h^2 + p(p-1)h^2\int_0^1 (1-t)(|x+th|^{p-2} - |x|^{p-2})dt. \tag{3.2}$$

Now, from the subadditivity of $u \to u^{p-2}$ on $[0, \infty[$ for p in $]2, 3]$, we get that

$$|x+th|^{p-2} - |x|^{p-2} \le (|x| + |th|)^{p-2} - |x|^{p-2} \le |th|^{p-2},$$

whence

$$p(p-1)h^2\int_0^1 (1-t)(|x+t|^{p-2} - |x|^{p-2})dt \le |h|^p \int_0^1 (t^{p-2} - t^{p-1})dt.$$

It follows that

$$\psi(x+h) - \psi(x) \le \psi'(x)h + \frac{1}{2}\psi''(x)h^2 + |h|^p. \tag{3.3}$$

Starting from (3.1), applying (3.3) and taking the expectation, we get that

$$\mathbb{E}(|S_{2^N}|^p) \le 2^N\mathbb{E}(|X_0|^p) + A_1 + (A_2/2) \tag{3.4}$$

where

$$A_1 = \sum_{n=1}^{2^N} \mathbb{E}(\psi'(S_{n-1})X_n), \quad \text{and} \quad A_2 = \sum_{n=1}^{2^N} \mathbb{E}\big(\psi''(S_{n-1})X_n^2\big).$$

In order to continue the decomposition , we will introduce a dyadic decomposition. Here we need further notation.

Notation 3.2 We set $n_0 = n - 1$ and we write n_0 in basis 2:

$$n_0 = b_N 2^N + b_{N-1}2^{N-1} + \cdots + b_0 2^0 \quad \text{with } b_i \in \{0, 1\}$$

(note that $b_N = 0$). Set $n_L = b_N 2^N + b_{N-1}2^{N-1} + \cdots + b_L 2^L$, so that $n_N = 0$. Let

$$I_{L,k} =]k2^L, (k+1)2^L], \qquad U_L^{(k)} = \sum_{i \in I_{L,k}} X_i, \; W_L^{(k)} = \sum_{i \in I_{L,k}} X_i^2 \quad \text{and}$$

$$V_L^{(k)} = E_{k2^L}(U_L^{(k)}),$$

where $E_j(Z) = \mathbb{E}(Z \mid \mathcal{F}_j)$. For sake of brevity write $U_L^{(0)} = U_L$ (thus $S_{2^N} = U_N$), $V_L^{(1)} = V_L$ and $M_p = \mathbb{E}(|X_0|^p)$.

From the elementary identity

$$f(S_{n-1}) = f(0) + \sum_{L=1}^{N} (f(S_{n_{L-1}}) - f(S_{n_L})) \tag{3.5}$$

applied to $f = \psi'$, we have

$$A_1 = \sum_{n=1}^{2^N} \sum_{L=1}^{N} \mathbb{E}((\psi'(S_{n_{L-1}}) - \psi'(S_{n_L}))X_n).$$

For k in $I_{N-L,0}$, we now sum on the integers n such that $n_L = (k-1)2^L$. Note that $n_{L-1} \neq n_L$ only if $b_{L-1} = 1$, which means that n belongs to $](2k-1)2^{L-1}, k2^L]$. Hence, interchanging the sum on n and the sums on k, we get that

$$A_1 = \sum_{L=1}^{N} \sum_{k=1}^{2^{N-L}} \mathbb{E}\Big(\psi'(S_{(2k-1)2^{L-1}}) - \psi'(S_{(k-1)2^L}) E_{(2k-1)2^{L-1}}(U_{L-1}^{(2k-1)})\Big). \tag{3.6}$$

Next, from the elementary inequality

$$|\psi'(x+h) - \psi'(x) - h\psi''(x)| \leq p|h|^{p-1} \tag{3.7}$$

applied to $x = S_{(k-1)2^L}$ and $h = S_{(2k-1)2^{L-1}} - S_{(k-1)2^L} = U_{L-1}^{(2k-2)}$, we infer that

$$|(\psi'(S_{(2k-1)2^{L-1}}) - \psi'(S_{(k-1)2^L}) - \psi''(S_{(k-1)2^L})U_{L-1}^{(2k-2)})| \leq p|U_{L-1}^{(2k-2)}|^{p-1}.$$

Multiplying the above inequality by $|V_{L-1}^{(2k-1)}|$, we obtain

$$A_1 \leq A_1' + p \sum_{L=1}^{N} \sum_{k=1}^{2^{N-L}} \mathbb{E}\Big(|U_{L-1}^{(2k-2)}|^{p-1} |V_{L-1}^{(2k-1)}|\Big),$$

where

$$A_1' = \sum_{L=1}^{N} \sum_{k=1}^{2^{N-L}} \mathbb{E}\Big(\psi''(S_{(k-1)2^L}) U_{L-1}^{(2k-2)} V_{L-1}^{(2k-1)}\Big). \tag{3.8}$$

From the stationarity of the sequence $(X_i)_{i\in\mathbb{Z}}$,

$$\mathbb{E}\left(\sum_{L\in[1,N]} \sum_{k\in I_{N-L,0}} |U_{L-1}^{(2k-2)}|^{p-1} |V_{L-1}^{(k)}|\right) = \sum_{L=0}^{N-1} 2^{N-L-1} \mathbb{E}\Big(|U_L|^{p-1}|V_L|\Big).$$

Together with the Hölder inequality

$$\mathbb{E}\big(|U_L|^{p-1}|V_L|\big) \leq \|U_L\|_p^{p-1} \|V_L\|_p,$$

it ensures that

$$A_1 \le A_1' + p2^{N-1}\sum_{L=0}^{N-1} 2^{-L}\|U_L\|_p^{p-1}\|V_L\|_p. \tag{3.9}$$

From (3.4) and (3.9), we now have to bound up $2A_1' + A_2$. Let

$$A_3 = 2\sum_{L=1}^{N}\sum_{k=1}^{2^{N-L}} \mathrm{Cov}\Big(\psi''(S_{(k-1)2^L}), U_{L-1}^{(2k-2)}V_{L-1}^{(2k-1)}\Big)$$

and $A_3' = A_1' - A_3$. We start by noting that

$$A_3 = \sum_{L=1}^{N}\sum_{k=1}^{2^{N-L}} \mathrm{Cov}\Big(\psi''(S_{(k-1)2^L}), U_{L-1}^{(2k-2)}U_{L-1}^{(2k-1)}\Big).$$

Next, writing $k_0 = k-1$ in basis 2 and using the same notations as for $n_0 = n-1$ (refer to Notation 3.2),

$$\psi''(S_{(k-1)2^L}) = \sum_{M=1}^{N-L}(\psi''(S_{k_{M-1}2^L}) - \psi''(S_{k_M 2^L})).$$

Setting $K = L + M$ and changing the order of summation, we then get

$$A_3 = \sum_{K=2}^{N}\sum_{k=1}^{2^{N-K}} \mathrm{Cov}\Bigg(\psi''(S_{(2k-1)2^{K-1}}) - \psi''(S_{(k-1)2^K}), \sum_{(L,l)} U_{L-1}^{(2l-2)}U_{L-1}^{(2l-1)}\Bigg), \tag{3.10}$$

where the sum on (L,l) ranges over all indices (L,l) such that $I_{L,l-1} \subset I_{K-1,2k}$. Now

$$W_{K-1}^{(2k-1)} + 2\sum_{(L,l)} U_{L-1}^{(2l-2)}U_{L-1}^{(2l-1)} = (U_{K-1}^{(2k-1)})^2,$$

which ensures that

$$2A_3 = \sum_{K=1}^{N}\sum_{k=1}^{2^{N-K}} \mathrm{Cov}\Big(\psi''(S_{(2k-1)2^{K-1}}) - \psi''(S_{(k-1)2^K}), (U_{K-1}^{(2k-1)})^2 - W_{K-1}^{(2k-1)}\Big).$$

We now decompose A_2 in the same way: $A_2 = A_4 + A_4'$, with

$$A_4 = \sum_{n=1}^{2^N} \mathrm{Cov}(\psi''(S_{n-1}), X_n^2).$$

Proceeding exactly as in the proof of (3.6), we get that

$$A_4 = \sum_{K=1}^{N}\sum_{k=1}^{2^{N-K}} \mathrm{Cov}\Big(\psi''(S_{(2k-1)2^{K-1}}) - \psi''(S_{(k-1)2^K}), W_{K-1}^{(2k-1)}\Big).$$

Hence

$$2A_3 + A_4 = \sum_{K=1}^{N} \sum_{k=1}^{2^{N-K}} \mathrm{Cov}\Big(\psi''(S_{(2k-1)2^{K-1}}) - \psi''(S_{(k-1)2^K}), (U_{K-1}^{(2k-1)})^2\Big)$$

$$= \sum_{K=1}^{N} \sum_{k=1}^{2^{N-K}} \mathbb{E}\Big((\psi''(S_{(2k-1)2^{K-1}}) - \psi''(S_{(k-1)2^K}))Z_{K-1}^{(2k-1)}\Big), \quad (3.11)$$

where

$$Z_{K-1}^{(l)} = E_{l2^{K-1}}(U_{K-1}^{(l)})^2 - \mathbb{E}((U_{K-1}^{(l)})^2). \quad (3.12)$$

Next, using the subadditivity of ψ'' and applying the Hölder inequality,

$$2A_3 + A_4 \le \sum_{K=1}^{N} \sum_{k=1}^{2^{N-K}} \mathbb{E}\Big(|\psi''(U_{K-1}^{(2k-2)})Z_{K-1}^{(2k-1)}|\Big)$$

$$\le p(p-1) \sum_{K=1}^{N} \sum_{k=1}^{2^{N-K}} \|U_{K-1}^{(2k-2)}\|_p^{p-2} \|Z_{K-1}^{(2k-1)}\|_{p/2}. \quad (3.13)$$

Setting $L = K - 1$, from the stationarity of $(X_i)_{i\in\mathbb{Z}}$ it follows that

$$2A_3 + A_4 \le p(p-1) \sum_{L=0}^{N-1} 2^{N-L-1} \|U_L\|_p^{p-2} \|Z_L\|_{p/2}, \quad \text{where } Z_L = Z_L^{(0)}. \quad (3.14)$$

It remains to bound up $2A_3' + A_4'$. From the stationarity of $(X_i)_{i\in\mathbb{Z}}$,

$$2A_3' + A_4' = \sum_{n=1}^{2^N} \mathbb{E}\big(\psi''(S_{n-1})\big)\mathbb{E}(X_0^2) + 2\sum_{L=1}^{N} \sum_{k=1}^{2^{N-L}} \mathbb{E}\big(\psi''(S_{(k-1)2^L})\big)\mathbb{E}(U_{L-1}V_{L-1}).$$

Next, by the Cauchy-Schwarz inequality, $\mathbb{E}(U_{L-1}V_{L-1}) \le \|U_{L-1}\|_2 \|V_{L-1}\|_2$. Hence, by (1.3),

$$\mathbb{E}(U_{L-1}V_{L-1}) \le 2^{(L-1)/2}\sigma_{L-1}\|V_{L-1}\|_2$$

with the convention that $\sigma_0 = \|X_0\|_2$. Since ψ'' is nonnegative, it follows that

$$2A_3' + A_4' \le \sum_{n=1}^{2^N} \mathbb{E}\big(\psi''(S_{n-1})\big)\sigma_0^2$$

$$+ 2\sum_{L=1}^{N} \sum_{k=1}^{2^{N-L}} \mathbb{E}\big(\psi''(S_{(k-1)2^L})\big)2^{(L-1)/2}\sigma_{L-1}\|V_{L-1}\|_2. \quad (3.15)$$

Now, for any $m < 2^N$,

$$\mathbb{E}\big(\psi''(S_m)\big) \le p(p-1)\|S_m\|_2^{(p-2)/2} \le p(p-1)\sigma_N^{p-2} m^{(p-2)/2}.$$

Hence

$$\sum_{n=1}^{2^N} \mathbb{E}\big(\psi''(S_{n-1})\big) \le p(p-1)\sigma_N^{p-2} \int_0^{2^N} x^{(p-2)/2} dx \le 2(p-1)\sigma_N^{p-2} 2^{Np/2}$$

and, in the same way,

$$\begin{aligned}\sum_{k=1}^{2^{N-L}} \mathbb{E}\big(\psi''(S_{(k-1)2^L})\big) &\le p(p-1)\sigma_N^{p-2} 2^{-L} \int_0^{2^N} x^{(p-2)/2} dx \\ &\le 2(p-1)\sigma_N^{p-2} 2^{-L+Np/2}.\end{aligned}$$

Setting $K = L - 1$, we then infer from the above inequalities that

$$2A_3' + A_4' \le 2(p-1)\sigma_N^{p-2} 2^{Np/2} \Bigg(\sigma_0^2 + \sum_{K=0}^{N-1} \sigma_K 2^{-K/2} \|V_K\|_2 \Bigg). \tag{3.16}$$

Now

$$\sigma_K 2^{-K/2} \|V_K\|_2 = 2\sigma_k(\sigma_{K+1} - \sigma_K) \le \sigma_{K+1}^2 - \sigma_K^2.$$

Hence, from (3.16),

$$2A_3' + A_4' \le 2(p-1)\sigma_N^p 2^{Np/2}. \tag{3.17}$$

Both (3.4), (3.9), (3.14) and (3.17) then yield the induction inequality below:

$$\begin{aligned}2^{-N} \|U_N\|_p^p &\le (p-1)\sigma_N^p 2^{N(p-2)/2} + M_p \\ &\quad + p \sum_{L=0}^{N-1} 2^{-1-L} \Bigg(\|U_L\|_p^{p-1} \|V_L\|_p + \frac{(p-1)}{2} \|U_L\|_p^{p-2} \|Z_L\|_{p/2} \Bigg),\end{aligned} \tag{3.18}$$

where the random variables Z_L are defined in (3.12) and (3.14).

Starting from (3.18), we now prove Theorem 3.1. Let $\zeta_N = 2^{-N/p} \|U_N\|_p$. Since $2^{-L/p} \|V_l\|_p = D_{L+1} - D_L$ and $2^{-2L/p} \|Z_L\|_{p/2} = \Delta_{L+1} - \Delta_L$ (here $D_0 = \Delta_0 = 0$), (3.18) is equivalent to

$$\begin{aligned}\zeta_N^p &\le (p-1)\sigma_N^p 2^{N(p-2)/2} + M_p \\ &\quad + \frac{p}{2} \sum_{L=0}^{N-1} \Bigg(\zeta_L^{p-1} (D_{L+1} - D_L) + \frac{(p-1)}{2} \zeta_L^{p-2} (\Delta_{L+1} - \Delta_L) \Bigg).\end{aligned} \tag{3.19}$$

From (3.19), we get immediately, by induction on N, that $\zeta_N \le \chi_N$ for any natural N, where $(\chi_N)_N$ is the sequence of positive reals defined by $\chi_0 = \zeta_0$ and

$$\chi_N^p = (p-1)\sigma_N^p 2^{N(p-2)/2} + M_p + \frac{p}{2}\sum_{L=0}^{N-1}\left(\chi_L^{p-1}(D_{L+1}-D_L) + \frac{(p-1)}{2}\chi_L^{p-2}(\Delta_{L+1}-\Delta_L)\right). \quad (3.20)$$

It follows from (3.20) that, for any natural integer, $\chi_{N+1}^p \geq \chi_N^p$. Hence the sequence (χ_L) is nondecreasing. Thus we have $\chi_L \leq \chi_N$ for any L in $[0, N-1]$. Since the reals $D_{L+1} - D_L$ and $\Delta_{L+1} - \Delta_L$ are nonnegative, it implies that

$$\sum_{L=0}^{N-1} \chi_L^{p-1}(D_{L+1}-D_L) \leq \chi_N^{p-1} D_N \quad \text{and} \quad \sum_{L=0}^{N-1} \chi_L^{p-2}(\Delta_{L+1}-\Delta_L) \leq \chi_N^{p-2}\Delta_N. \quad (3.21)$$

Hence

$$\chi_N^p \leq (p-1)\sigma_N^p 2^{N(p-2)/2} + M_p + \frac{p}{2}\left(\chi_N^{p-1} D_N + \frac{(p-1)}{2}\chi_N^{p-2}\Delta_N\right). \quad (3.22)$$

Now, if $D_N < \chi_N/(2p)$ and $\Delta_N < \chi_N^2/(p^2-p)$, then, by (3.22),

$$\chi_N^p \leq 2(p-1)\sigma_N^p 2^{N(p-2)/2} + 2M_p.$$

Hence

$$\chi_N^p \leq \max(2(p-1)\sigma_N^p 2^{N(p-2)/2} + 2M_p, (2pD_N)^p, (p(p-1)\Delta_N)^{p/2}),$$

which implies Theorem 3.1 in the case $n = 2^N$.

If $n > 2^N$, then $n = \sum_{L=0}^N b_L 2^L$ with $b_L = 0$ or $b_L = 1$, and $b_N = 1$. Then, by stationarity of $(X_i)_{i\in\mathbb{Z}}$,

$$\begin{aligned}\|S_n\|_p &\leq \sum_{L=0}^N b_L \|U_L\|_p \\ &\leq \sum_{L=0}^N b_L 2^{L/p} \max(a_p\sigma_N^p 2^{N(p-2)/2} + 2M_p, (c_p\Delta_N)^{p/2}, (d_p D_N)^p)\end{aligned}$$

using Theorem 3.1 in the case $n = 2^L$ and the monotonicity of the above sequences. Theorem 3.1 follows then from the fact that

$$\sum_{L=0}^N b_L 2^{L/p} \leq 2^{N/p}/(1-2^{-1/p}). \qquad \square$$

4 Application to an Example of Irreducible Markov Chain

In this section we apply the results of Sect. 3 to a Markov chain which is a symmetrized version of the Harris recurrent Markov chain defined in [7]. Let $E = [-1, 1]$

and ν be a symmetric atomless law on E. The Markov kernel K is defined by

$$K(x,.) = (1-|x|)\delta_x + |x|\nu,$$

where δ_x denotes the Dirac measure at point x. If

$$\int_E |x|^{-1} d\nu(x) < \infty, \tag{4.1}$$

then there exists an unique invariant probability measure π and

$$\pi = \left(\int_E |x|^{-1} d\nu(x)\right)^{-1} |x|^{-1}\nu.$$

Then the stationary chain $(\xi_i)_{i\in\mathbb{Z}}$ with kernel K is positively recurrent. We refer to Lemma 2, in [7, p. 75] or to [1] for estimates of the absolute regularity coefficients of the stationary chain with kernel K (see also [11] for more about the connections between regularity and ergodicity).

Let f be a measurable function on E and $X_i = f(\xi_i)$. We denote by $S_n(f)$ the partial sum S_n attached to the r.v.'s X_i. Our aim is to apply the moment inequalities of Sect. 3 to $S_n(f)$. In order to get more tractable estimates, we will assume throughout Sect. 4 that the function f is odd.

Proposition 4.1 *Let f be a function satisfying $f(-x) = -f(x)$ for any x in E.*

(a) *Let δ be some real in* $]0,1[$ *and p be any real in* $[1,\infty]$. *Assume that, for some positive t, $|f| \le g$ on $[-t,t]$ for some even function g on E such that g is nondecreasing on* $[0,1]$ *and $x^{-1}g$ is nonincreasing on* $[0,1]$ *and*

$$\int_0^1 x^{-2+\delta} g(x)dx < \infty. \tag{4.2}$$

If f belongs to $L^p(\pi)$, then $\sum_{N>0} 2^{-N\delta}\|E_0(S_n(f))\|_p < \infty$.

(b) *Let p be any real in* $]2,3]$. *Assume that $|f(x)| \le Cx^{1/2}$ for any x in E and that the measure ν satisfies $\nu([0,t]) \le ct^{a+1}$ for some $a > (p-2)/2$ and some positive constant c. Then*

$$\sum_{N>0} 2^{-N/p}\|E_0(S_n(f))\|_p < \infty \quad \textit{and}$$

$$\sum_{N>0} 2^{-2N/p}\|E_0(S_n^2(f)) - \mathbb{E}(S_n^2(f))\|_{p/2} < \infty.$$

Remark 4.1 Applying (a) with $\delta = 1/2$ and $p \ge 2$ we get that $S_n(f)$ satisfies the Marcinkiewicz-Zygmund inequality of [12] for any f in $L^p(\pi)$ verifying (4.2) and the above local monotonicity conditions Furthermore, for bounded functions f satisfying these monotonicity conditions, the Hoeffding inequality in [12] holds true under (4.2) with $\delta = 1/2$. In particular if f is bounded and satisfies $f(x) = O(x^{1/2}(\log(e/x))^{-1-\varepsilon})$ as x tends to 0, then the Hoeffding inequality holds. By contrast the charge condition in [5] needs the too restrictive condition $f(x) = O(x)$.

For p in $]2,3]$, Theorem 3.1 applies under the conditions of Proposition 4.1(b). if ν has the density $(a+1)|x|^a/2$ with respect to the Lebesgue measure, then the absolute regularity coefficients β_n (they are defined in (4.8) below) of the chain $(\xi_i)_i$ are exactly of the order of n^{-a}, as shown by Lemma 2, in [7, p. 75]. Consequently the Rosenthal inequality holds for functions f satisfying the condition of Proposition 4.1(b), even if the Markov chain does not satisfy the condition $\sum_n \beta_n < \infty$, which corresponds to the ergodicity of degree two, as described in [11].

Proof of Proposition 4.1 Throughout the proof, $\mathbb{E}_x$ denotes the probability of the chain with kernel K starting from $\xi_0 = x$. Let $\tau = \inf\{k \geq 0 : \xi_k \neq \xi_0\}$. Now

$$\mathbb{E}_x(S_n(f)) = \mathbb{E}_x\left(nf(x)\mathbb{1}_{\tau>n} + \sum_{k=1}^{n}\big(\mathbb{E}_\nu(f(\xi_0) + S_{n-k}(f)) + (k-1)f(x)\big)\mathbb{1}_{\tau=k}\right).$$

From the definition of K and the symmetry of ν we easily get that νK^l is a symmetric law for any positive integer l, whence $\mathbb{E}_\nu(f(\xi_0) + S_{n-k}(f)) = 0$ for any odd function f. Now $\mathbb{E}_x(g(\xi_n)) = (1-|x|)^n g(x)$ for any odd function g, which implies that

$$\mathbb{E}_x(S_n(f)) = \mathbb{E}_x(\min(\tau-1, n))f(x) = (1-|x|)(1-(1-|x|)^n)|x|^{-1}f(x). \tag{4.3}$$

Since $1-(1-|x|)^n \leq \min(n|x|, 1)$, it follows that

$$n^{-1}|\mathbb{E}_x(S_n(f))| \leq (\max(n|x|, 1))^{-1}|f(x)| \quad \text{for any } x \in E. \tag{4.4}$$

Hence, for any $n \geq 1/t$,

$$(\max(n|x|, 1))^{-1}|f(x)| \leq g(1/n) \quad \text{for any } x \in [-t, t]. \tag{4.5}$$

From (4.4) and (4.5) we get that, for $n \geq 1/t$,

$$|\mathbb{E}_x(S_n(f))| \leq ng(1/n) + t^{-1}|f(x)| \quad \text{for any } x \in [-1, 1]. \tag{4.6}$$

Hence, for any δ in $]0,1[$ and any p in $[0,\infty]$, the series $\sum_N 2^{-N\delta}\|E_0(S_{2^N})\|_p$ is convergent if f belongs to $L^p(\pi)$ and

$$\sum_{N>0} 2^{N(1-\delta)}g(2^{-N}) < \infty, \tag{4.7}$$

(here $\log_2$ denotes the logarithm in basis 2), which is equivalent to (4.2).

We now prove Proposition 4.1(b). Since $(p-2)/2 \leq 1/2$ we may assume $a \leq 1$. By Lemma 2, in [7, p. 75],

$$\beta_n = \int_E \|\delta_x K^n - \pi\| d\pi(x) = O(n^{-a}) \quad \text{as } n \to \infty, \tag{4.8}$$

where $\|\mu\|$ denotes the total variation of the signed measure μ. We now give estimates of the $L^{p/2}$-norm of the random variables $E_0(S_n^2) - \mathbb{E}(S_n^2)$. Writing

$$S_n^2(f) = \sum_{l=1}^{n} f(\xi_l)\left(f(\xi_l) + 2\sum_{m=l+1}^{n} f(\xi_m)\right), \tag{4.9}$$

we get that

$$\mathbb{E}_x(S_n^2(f)) = \sum_{l=1}^{n} \delta_x K^l \left(f^2 + 2f \sum_{m=1}^{n-l} K^m f \right). \tag{4.10}$$

Hence

$$\mathbb{E}(S_n^2) = \int_E \mathbb{E}_x(S_n^2(f)) d\pi(x) = \sum_{l=1}^{n} \pi \left(f^2 + 2f \sum_{m=1}^{n-l} K^m f \right), \tag{4.11}$$

using the fact that $\pi K^l = \pi$. From the above equalities, we infer that

$$\| E_0(S_n^2) - \mathbb{E}(S_n^2) \|_{p/2} \le \sum_{l=1}^{n} \left(\int_E \left| (\delta_x K^l - \pi) \left(f^2 + 2f \sum_{m=1}^{n-l} K^m f \right) \right|^{p/2} d\pi(x) \right)^{2/p}. \tag{4.12}$$

Now, by (4.3),

$$\left(f^2 + 2f \sum_{m=1}^{n-l} K^m f \right)(x) = f^2(x) \big(1 + 2(1 - (1 - |x|)^{n-l})(|x|^{-1} - 1) \big), \tag{4.13}$$

which ensures that

$$\sup_{x \in E} \left| f^2(x) + 2f(x) \sum_{m=1}^{n-l} K^m f(x) \right| \le 2 \sup_{x \in E} |x|^{-1} f^2(x) \le 2C^2$$

under the assumptions of Proposition 4.1(b). Hence

$$\begin{aligned} &\left| (\delta_x K^l - \pi) \left(f^2 + 2f \sum_{m=1}^{n-l} K^m f \right) \right|^{p/2} \\ &\quad \le (2C)^{p-2} \left| (\delta_x K^l - \pi) \left(f^2 + 2f \sum_{m=1}^{n-l} K^m f \right) \right| \\ &\quad \le 2^{p-1} C^p \| \delta_x K^l - \pi \|. \end{aligned} \tag{4.14}$$

Both (4.12) and (4.14) imply that

$$\begin{aligned} \| E_0(S_n^2) - \mathbb{E}(S_n^2) \|_{p/2} &\le 4C^2 \sum_{l=1}^{n} \left(\int_E \| \delta_x K^l - \pi \| d\pi(x) \right)^{2/p} \\ &\le C' \sum_{l=1}^{n} l^{-2a/p} \end{aligned} \tag{4.15}$$

by (4.8). Since $2a/p < 1$, (4.15) entails that

$$2^{-2N/p} \| E_0(S_{2^N}^2) - \mathbb{E}(S_{2^N}^2) \|_{p/2} = O(2^{N(1-2(a+1)/p)}) \quad \text{as } N \to \infty.$$

Therefrom Δ_N converges to some finite Δ as N tends to ∞ as soon as $a > (p-2)/2$.

It remains to prove that D_N converges to some finite real as soon as $a > (p-2)/2$. By (4.3)

$$\|E_0(S_n)\|_p^p \le 2C^p\left(\int_0^{1/n} n^p x^{p/2} d\pi(x) + \int_{1/n}^1 x^{-p/2} d\pi(x)\right).$$

Recall that $\pi = c' x^{-1}\nu$ for some positive constant c'. Hence

$$\begin{aligned}\int_0^{1/n} n^p x^{p/2} d\pi(x) &= c' \int_0^{1/n} n^p x^{(p-2)/2} d\pi(x)\\ &\le c' n^{1+p/2}\nu([0, 1/n]) \le c' c n^{-a+p/2}.\end{aligned}$$

In the same way, integrating by parts, we get that

$$\int_{1/n}^1 x^{-p/2} d\pi(x) = c' \int_{1/n}^1 x^{-1-p/2} d\nu(x) \le C' n^{-a+p/2}$$

for some positive constant C'. Hence

$$2^{-N/p}\|E_0(S_{2^N})\|_p = O(2^{N(p-2-2a)/2p}) \quad \text{as } N \to \infty,$$

which implies the convergence of D_N as soon as $a > (p-2)/2$. □

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