MLP starting ideas

July 15, 2022

Abstract

Abstract goes here...

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2 Multilevel Picard approximations for the heat equation

Theorem 2.1. *Let* $T, \kappa, \delta \in (0, \infty)$, $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$, *let* $u_d \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, *satisfy for all* $d \in \mathbb{N}$, $t \in [0,T]$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ *that*

$$|u_d(t,x)| \leq \kappa d^\kappa \bigl(1 + \textstyle\sum_{k=1}^d |x_k|\bigr)^\kappa \qquad \text{and} \qquad (\tfrac{\partial}{\partial t} u_d)(t,x) = (\Delta_x u_d)(t,x), \tag{2.1}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $W^{d,\theta} \colon [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\theta \in \Theta$, *be independent standard Brownian motions, let* $U_m^{d,\theta} \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $d, m \in \mathbb{Z}$, $\theta \in \Theta$, *satisfy for all* $d, m \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ *that*

$$U_m^{d,\theta}(t,x) = \frac{1}{m} \Biggl[\sum_{k=1}^m u_d\bigl(0, x + \sqrt{2}\, W_t^{d,(\theta,0,-k)}\bigr) \Biggr],$$

and for every $d, n, m \in \mathbb{N}$ *let* $\mathfrak{C}_{d,n,m} \in \mathbb{N}$ *be the number of function evaluations of* $u_d(0,\cdot)$ *and the number of realizations of scalar random variables which are used to compute one realization of* $U_m^{d,0}(T,0) \colon \Omega \to \mathbb{R}$. *Then there exist* $c \in \mathbb{R}$ *and* $\mathfrak{n} \colon \mathbb{N} \times (0,1] \to \mathbb{N}$ *such that for all* $d \in \mathbb{N}$, $\varepsilon \in (0,1]$ *it holds that*

$$\Bigl(\mathbb{E}\bigl[|u_d(T,0) - U_{\mathfrak{n}(d,\varepsilon)}^{d,0}(T,0)|^2 \bigr] \Bigr)^{1/2} \leq \varepsilon \qquad \text{and} \qquad \mathfrak{C}_{d,\mathfrak{n}(d,\varepsilon),\mathfrak{n}(d,\varepsilon)} \leq c d^c \varepsilon^{-(2+\delta)}. \tag{2.2}$$

3 Stochastic solutions to parabolic partial differential equations

Lemma 3.1. *Let $T \in (0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $u_d \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, $d\in\mathbb{N}$, satisfy for all $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that*

$$(\tfrac{\partial}{\partial t}u_d)(t,x) + (\Delta_x u_d)(t,x) = 0, \tag{3.1}$$

let $W^d\colon [0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, be standard Brownian motions, and let $\mathcal{X}^{d,t,x}\colon [t,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$, be a stochastic process with continuous sample paths satisfying that for all $d\in\mathbb{N}$, $t\in[0,T]$, $s\in[t,T]$, $x\in\mathbb{R}^d$ we have $\mathbb{P}$-a.s. that

$$\mathcal{X}^{d,t,x}_s = x + \int_t^s \sqrt{2}\,\mathrm{d}W^d_r = x + \sqrt{2}\,W^d_{t-s}. \tag{3.2}$$

Then for all $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$u_d(t,x) = \mathbb{E}\Big[u_d\big(T,\mathcal{X}^{d,t,x}_T\big)\Big]. \tag{3.3}$$

Proof of Lemma 3.1. The proof of Lemma 3.1 is thus complete. □

Lemma 3.2. *Let $T \in (0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\sigma_d\colon\mathbb{R}^d\to\mathbb{R}^{d\times d}$, $d\in\mathbb{N}$, be infinitely often differentiable functions, let $u_d \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, $d\in\mathbb{N}$, satisfy for all $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that*

$$(\tfrac{\partial}{\partial t}u_d)(t,x) + \operatorname{Trace}\Big(\sigma(x)[\sigma(x)]^*(\operatorname{Hess}_x u_d)(t,x)\Big) = 0, \tag{3.4}$$

let $W^d\colon [0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, be standard Brownian motions, and let $\mathcal{X}^{d,t,x}\colon [t,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$, be a stochastic process with continuous sample paths satisfying that for all $d\in\mathbb{N}$, $t\in[0,T]$, $s\in[t,T]$, $x\in\mathbb{R}^d$ we have $\mathbb{P}$-a.s. that

$$\mathcal{X}^{d,t,x}_s = x + \int_s^t \sqrt{2}\,\sigma(\mathcal{X}^{d,t,x}_r)\,\mathrm{d}W^d_r. \tag{3.5}$$

Then for all $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$u_d(t,x) = \mathbb{E}\Big[u_d\big(T,\mathcal{X}^{d,t,x}_T\big)\Big]. \tag{3.6}$$

Proof of Lemma 3.2. The proof of Lemma 3.2 is thus complete. □

Lemma 3.3. *Let $T \in (0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\mu_d\in\mathbb{R}^d\to\mathbb{R}^d$, $d\in\mathbb{N}$, be infinitely often differentiable functions, let $u_d \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, $d\in\mathbb{N}$, satisfy for all $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that*

$$(\tfrac{\partial}{\partial t}u_d)(t,x) + (\Delta_x u_d)(t,x) + [\mu_d(x)]^*(\nabla_x u_d)(t,x) = 0, \tag{3.7}$$

let $W^d\colon [0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, be standard Brownian motions, and let $\mathcal{X}^{d,t,x}\colon [t,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$, be a stochastic process with continuous sample paths satisfying that for all $d\in\mathbb{N}$, $t\in[0,T]$, $s\in[t,T]$, $x\in\mathbb{R}^d$ we have $\mathbb{P}$-a.s. that

$$\mathcal{X}^{d,t,x}_s = x + \int_s^t \mu_d(\mathcal{X}^{d,t,x}_r)\,\mathrm{d}r + \int_s^t \sqrt{2}\,\mathrm{d}W^d_r. \tag{3.8}$$

Then for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$u_d(t,x) = \mathbb{E}\Big[u_d\big(T, \mathcal{X}_T^{d,t,x}\big)\Big]. \tag{3.9}$$

Proof of Lemma 3.3. The proof of Lemma 3.3 is thus complete. □

Lemma 3.4. *Let $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\alpha_d \in \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, be infinitely often differentiable functions, let $u_d \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that*

$$(\tfrac{\partial}{\partial t} u_d)(t,x) + (\Delta_x u_d)(t,x) + \alpha_d(x) u_d(t,x) = 0, \tag{3.10}$$

let $W^d \colon [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, be standard Brownian motions, and let $\mathcal{X}^{d,t,x} \colon [t,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, be a stochastic process with continuous sample paths satisfying that for all $d \in \mathbb{N}$, $t \in [0,T]$, $s \in [t,T]$, $x \in \mathbb{R}^d$ we have $\mathbb{P}$-a.s. that

$$\mathcal{X}_s^{d,t,x} = x + \int_s^t \sqrt{2}\, \mathrm{d}W_r^d. \tag{3.11}$$

Then for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$u_d(t,x) = \mathbb{E}\Big[\exp\big(\textstyle\int_t^T \alpha_d(\mathcal{X}_r^{d,t,x})\,\mathrm{d}r\big) u_d\big(T, \mathcal{X}_T^{d,t,x}\big)\Big]. \tag{3.12}$$

Proof of Lemma 3.4. The proof of Lemma 3.4 is thus complete. □