Full error analysis for the training of deep neural networks

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Abstract

Deep learning algorithms have been applied very successfully in recent years to a range of problems out of reach for classical solution paradigms. Nevertheless, there is no completely rigorous mathematical error and convergence analysis which explains the success of deep learning algorithms. The error of a deep learning algorithm can in many situations be decomposed into three parts, the approximation error, the generalization error, and the optimization error. In this work we estimate for a certain deep learning algorithm each of these three errors and combine these three error estimates to obtain an overall error analysis for the deep learning algorithm under consideration. In particular, we thereby establish convergence with a suitable convergence speed for the overall error of the deep learning algorithm under consideration. Our convergence speed analysis is far from optimal and the convergence speed that we establish is rather slow, increases exponentially in the dimensions, and, in particular, suffers from the curse of dimensionality. The main contribution of this work is, instead, to provide a full error analysis (i) which covers each of the three different sources of errors usually emerging in deep learning algorithms and (ii) which merges these three sources of errors into one overall error estimate for the considered deep learning algorithm.

Contents

1 Introduction

In problems like image recognition, text analysis, speech recognition, or playing various games, to name a few, it is very hard and seems at the moment entirely impossible to provide a function or to hard-code a computer program which attaches to the input – be it a picture, a piece of text, an audio recording, or a certain game situation – a meaning or a recommended action. Nevertheless deep learning has been applied very successfully in recent years to such and related problems. The success of deep learning in applications is even more surprising as, to this day, the reasons for its performance are not entirely rigorously understood. In particular, there is no rigorous mathematical error and convergence analysis which explains the success of deep learning algorithms.

In contrast to traditional approaches, machine learning methods in general and deep learning methods in particular attempt to infer the unknown target function or at least a good enough approximation thereof from examples encountered during the training. Often a deep learning algorithm has three ingredients: (i) the *hypothesis class*, a parametrizable class of functions in which we try to find a reasonable approximation of the unknown target function, (ii) a *numerical approximation of the expected loss function* based on the training examples, and (iii) an *optimization algorithm* which tries to approximately calculate an element of the hypothesis class which minimizes the numerical approximation of the expected loss function from (ii) given the training examples. Common approaches are to choose a set of suitable fully connected deep neural networks (DNNs) as hypothesis class in (i), empirical risks as approximations of the expected loss function in (ii), and stochastic gradient descent-type algorithms with random initializations as optimization algorithms in (iii). Each of these three ingredients contributes to the overall error of the considered approximation algorithm. The choice of the hypothesis class results in the so-called *approximation error* (cf., e.g., [3, 4, 19, 38, 40, 41] and the references mentioned at the beginning of Section 3), replacing the exact expected loss function by a numerical approximation leads to the so-called *generalization error* (cf., e.g., [5, 10, 18, 35, 51, 68, 71] and the references mentioned therein), and the employed optimization algorithm introduces the *optimization error* (cf., e.g., [2, 6, 9, 15, 20, 26, 43, 45] and the references mentioned therein).

In this work we estimate the approximation error, the generalization error, as well as the optimization error and we also combine these three errors to establish convergence with a suitable convergence speed for the overall error of the deep learning algorithm under consideration. Our convergence speed analysis is far from optimal and the convergence speed that we establish is rather slow, increases exponentially in the dimensions, and, in particular, suffers from the curse of dimensionality (cf., e.g., Bellman [8], Novak & Woźniakowski [56, Chapter 1], and Novak & Woźniakowski [57, Chapter 9]). The main contribution of this work is, instead, to provide a full error analysis (i) which covers each of the three different sources of errors usually emerging in deep learning algorithms and (ii) which merges these three sources of errors into one overall error estimate for the considered deep learning algorithm. In the next result, Theorem 1.1, we briefly illustrate the findings of this article in a special case and we refer to Section 4.2 below for the more general convergence results which we develop in this article.

Theorem 1.1. *Let $d \in \mathbb{N}$, $L, a \in \mathbb{R}$, $b \in (a, \infty)$, $R \in [\max\{2, L, |a|, |b|\}, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X_m \colon \Omega \to [a, b]^d$, $m \in \mathbb{N}$, be i.i.d. random variables, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $\varphi \colon [a, b]^d \to [0, 1]$ satisfy for all $x, y \in [a, b]^d$ that $|\varphi(x) - \varphi(y)| \leq L\|x - y\|$, for every $\mathfrak{d}, r, s \in \mathbb{N}$, $\delta \in \mathbb{N}_0$, $\theta = (\theta_1, \theta_2, \ldots, \theta_{\mathfrak{d}}) \in \mathbb{R}^{\mathfrak{d}}$ with $\mathfrak{d} \geq \delta + rs + r$ let $\mathcal{A}^{\theta,\delta}_{r,s} \colon \mathbb{R}^s \to \mathbb{R}^r$ satisfy for all $x = (x_1, x_2, \ldots, x_s) \in \mathbb{R}^s$ that*

$$\mathcal{A}^{\theta,\delta}_{r,s}(x) = \left(\left[\sum_{i=1}^{s} x_i \theta_{\delta+i} \right] + \theta_{\delta+rs+1}, \left[\sum_{i=1}^{s} x_i \theta_{\delta+s+i} \right] + \theta_{\delta+rs+2}, \ldots, \left[\sum_{i=1}^{s} x_i \theta_{\delta+(r-1)s+i} \right] + \theta_{\delta+rs+r} \right), \tag{1}$$

let $\mathfrak{c} \colon \mathbb{R} \to [0, 1]$ and $\mathfrak{R}_\tau \colon \mathbb{R}^\tau \to \mathbb{R}^\tau$, $\tau \in \mathbb{N}$, satisfy for all $\tau \in \mathbb{N}$, $x = (x_1, x_2, \ldots, x_\tau) \in \mathbb{R}^\tau$, $y \in \mathbb{R}$ that $\mathfrak{c}(y) = \min\{1, \max\{0, y\}\}$ and $\mathfrak{R}_\tau(x) = (\max\{x_1, 0\}, \max\{x_2, 0\}, \ldots, \max\{x_\tau, 0\})$, for every $\mathfrak{d}, \tau \in \{3, 4, \ldots\}$, $\theta \in \mathbb{R}^{\mathfrak{d}}$ with $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ let $\mathfrak{N}^{\theta,\tau} \colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$ that

$$\big(\mathfrak{N}^{\theta,\tau}\big)(x) = \big(\mathfrak{c} \circ \mathcal{A}^{\theta,\tau(d+1)+(\tau-3)\tau(\tau+1)}_{1,\tau} \circ \mathfrak{R}_\tau \circ \mathcal{A}^{\theta,\tau(d+1)+(\tau-4)\tau(\tau+1)}_{\tau,\tau} \circ \mathfrak{R}_\tau \circ \ldots \circ \mathcal{A}^{\theta,\tau(d+1)}_{\tau,\tau} \circ \mathfrak{R}_\tau \circ \mathcal{A}^{\theta,0}_{\tau,d}\big)(x), \tag{2}$$

let $\mathfrak{E}_{\mathfrak{d},M,\tau} \colon [-R, R]^{\mathfrak{d}} \times \Omega \to [0, \infty)$, $\mathfrak{d}, M, \tau \in \mathbb{N}$, satisfy for all $\mathfrak{d}, M \in \mathbb{N}$, $\tau \in \{3, 4, \ldots\}$, $\theta \in [-R, R]^{\mathfrak{d}}$, $\omega \in \Omega$ with $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ that

$$\mathfrak{E}_{\mathfrak{d},M,\tau}(\theta, \omega) = \frac{1}{M} \left[\sum_{m=1}^{M} |\mathfrak{N}^{\theta,\tau}(X_m(\omega)) - \varphi(X_m(\omega))|^2 \right], \tag{3}$$

for every $\mathfrak{d} \in \mathbb{N}$ let $\Theta_{\mathfrak{d},k} \colon \Omega \to [-R, R]^{\mathfrak{d}}$, $k \in \mathbb{N}$, be i.i.d. random variables, assume for all $\mathfrak{d} \in \mathbb{N}$ that $\Theta_{\mathfrak{d},1}$ is continuous uniformly distributed on $[-R, R]^{\mathfrak{d}}$, and let $\Xi_{\mathfrak{d},K,M,\tau} \colon \Omega \to [-R, R]^{\mathfrak{d}}$, $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, satisfy for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$ that $\Xi_{\mathfrak{d},K,M,\tau} = \Theta_{\mathfrak{d},\min\{k \in \{1,2,\ldots,K\} \colon \mathfrak{E}_{\mathfrak{d},M,\tau}(\Theta_{\mathfrak{d},k}) = \min_{l \in \{1,2,\ldots,K\}} \mathfrak{E}_{\mathfrak{d},M,\tau}(\Theta_{\mathfrak{d},l})\}}$. Then there exists $c \in (0, \infty)$ such that for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, $\varepsilon \in (0, 1]$ with $\tau \geq 2d(2dL\varepsilon^{-1} + 2)^d$ and $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ it holds that

$$\mathbb{P}\left(\int_{[a,b]^d} |\mathfrak{N}^{\Xi_{\mathfrak{d},K,M,\tau},\tau}(x) - \varphi(x)| \, \mathbb{P}_{X_1}(dx) > \varepsilon \right) \leq \exp\big(-K(c\tau)^{-\tau\mathfrak{d}} \varepsilon^{2\mathfrak{d}}\big) + 2\exp\big(\mathfrak{d} \ln\big((c\tau)^\tau \varepsilon^{-2}\big) - \tfrac{\varepsilon^4 M}{c}\big). \tag{4}$$

Theorem 1.1 is an immediate consequence of Corollary 4.8 in Section 4.2 below. Corollary 4.8 follows from Corollary 4.7 which, in turn, is implied by Theorem 4.5, the main result of this article. In the following we add some comments and explanations regarding the mathematical objects which appear in Theorem 1.1 above. For every $\mathfrak{d}, \tau \in \{3, 4, ...\}, \theta \in \mathbb{R}^{\mathfrak{d}}$ with $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ the function $\mathfrak{N}^{\theta,\tau} \colon \mathbb{R}^{\mathfrak{d}} \to \mathbb{R}$ in (2) above describes the realization of a fully connected deep neural network with τ layers (1 input layer with d neurons [d dimensions], 1 output layer with 1 neuron [1 dimension], as well as $\tau - 2$ hidden layers with τ neurons on each hidden layer [τ dimensions in each hidden layer]). The vector $\theta \in \mathbb{R}^{\mathfrak{d}}$ in (2) in Theorem 1.1 above stores the real parameters (the weights and the biases) for the concrete considered neural network. In particular, the architecture of the deep neural network in (2) is chosen so that we have $\tau d + (\tau-3)\tau^2 + \tau$ real parameters in the weight matrices and $(\tau-2)\tau + 1$ real parameters in the bias vectors resulting in $[\tau d + (\tau-3)\tau^2 + \tau] + [(\tau-2)\tau + 1] = \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ real parameters for the deep neural network overall. This explains why the dimension $\mathfrak{d}$ of the parameter vector $\theta \in \mathbb{R}^{\mathfrak{d}}$ must be larger or equal than the number of real parameters used to describe the deep neural network in (2) in the sense that $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ (see above (2)). The affine linear transformations for the deep neural network, which appear just after the input layer and just after each hidden layer in (2), are specified in (1) above. The functions $\mathfrak{R}_\tau \colon \mathbb{R}^\tau \to \mathbb{R}$, $\tau \in \mathbb{N}$, describe the multi-dimensional rectifier functions which are employed as activation functions in (2). Realizations of the random variables $(X_m, Y_m) := (X_m, \varphi(X_m))$, $m \in \{1, 2, \ldots, M\}$, act as training data and the neural network parameter vector $\theta \in \mathbb{R}^{\mathfrak{d}}$ should be chosen so that the empirical risk in (3) gets minimized. In Theorem 1.1 above, we use as an optimization algorithm just random initializations and perform no gradient descent steps. The inequality in (4) in Theorem 1.1 above provides a quantitative error estimate for the probability that the L^1-distance between the trained deep neural network approximation $\mathfrak{N}^{\Xi_{\mathfrak{d},K,M,\tau},\tau}(x)$, $x \in [a,b]^d$, and the function $\varphi(x)$, $x \in [a,b]^d$, which we actually want to learn, is larger than a possibly arbitrarily small real number $\varepsilon \in (0,1]$. In (4) in Theorem 1.1 above we measure the error between the deep neural network and the function $\varphi \colon [a,b]^d \to [0,1]$, which we intend to learn, in the L^1-distance instead of in the L^2-distance. However, in the more general results in Section 4.2 below we measure the error in the L^2-distance and, just to keep the statement in Theorem 1.1 as easily accessible as possible, we restrict ourselves in Theorem 1.1 above to the L^1-distance. Observe that for every $\varepsilon \in (0,1]$ and every $\mathfrak{d}, \tau \in \{3, 4, \ldots\}$ with $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ we have that the right hand side of (4) converges to zero as K and M tend to infinity. The right hand side of (4) also specifies a concrete speed of convergence and in this sense Theorem 1.1 provides a full error analysis for the deep learning algorithm under consideration. Our analysis is in parts inspired by Maggi [50], Berner et al. [10], Cucker & Smale [18], Beck et al. [6], and Fehrman et al. [26].

The remainder of this article is organized as follows. In Section 2 we present two elementary approaches how DNNs can be described in a mathematical fashion. Both approaches will be used in our error analyses in the later parts of this article. In Section 3 we separately analyze the approximation error, the generalization error, and the optimization error of the considered algorithm. In Section 4 we combine the separate error analyses in Section 3 to obtain an overall error analysis of the considered algorithm.

2 Deep neural networks (DNNs)

In this section we present two elementary approaches on how DNNs can be described in a mathematical fashion. More specifically, we present in Section 2.1 a vectorized description for DNNs and we present in Section 2.2 a structured description for DNNs. Both approaches will be used in our error analyses in the later parts of this article. Sections 2.1, 2.2, and 2.3 are partially based on material in publications from the scientific literature such as Beck et al. [6, 7], Berner et al. [10], Goodfellow et al. [28], and Grohs et al. [31, 32]. In particular, Definition 2.1 is inspired by, e.g., (25) in [7], Definition 2.2 is inspired by, e.g., (26) in [7], Definition 2.3 is, e.g., [31, Definition 2.2], Definitions 2.4, 2.5, 2.6, 2.7, and 2.8 are inspired by, e.g., [10, Setting 2.3], Definition 2.9 is, e.g., [31, Definition 2.1], Definition 2.10 is, e.g., [31, Definition 2.3], Definition 2.16 is, e.g., [31, Definition 2.17], Definition 2.17 is, e.g., [32, Definition 3.10], Definition 2.18 is, e.g., [32, Definition 3.15], Definition 2.19 is, e.g., [31, Definition 2.5], Definition 2.23 is, e.g., [31, Definition 2.11], Definition 2.24 is, e.g., [31, Definition 2.12], and Theorem 2.36 is a strengthened version of [10, Theorem 4.2].

2.1 Vectorized description of DNNs

2.1.1 Affine functions

Definition 2.1 (Affine function)**.** Let $d, r, s \in \mathbb{N}$, $\delta \in \mathbb{N}_0$, $\theta = (\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d$ satisfy $d \geq \delta + rs + r$. Then we denote by $\mathcal{A}^{\theta,\delta}_{r,s} \colon \mathbb{R}^s \to \mathbb{R}^r$ the function which satisfies for all $x = (x_1, x_2, \ldots, x_s) \in \mathbb{R}^s$ that

$$\mathcal{A}^{\theta,\delta}_{r,s}(x) = \begin{pmatrix} \theta_{\delta+1} & \theta_{\delta+2} & \cdots & \theta_{\delta+s} \\ \theta_{\delta+s+1} & \theta_{\delta+s+2} & \cdots & \theta_{\delta+2s} \\ \theta_{\delta+2s+1} & \theta_{\delta+2s+2} & \cdots & \theta_{\delta+3s} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{\delta+(r-1)s+1} & \theta_{\delta+(r-1)s+2} & \cdots & \theta_{\delta+rs} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_s \end{pmatrix} + \begin{pmatrix} \theta_{\delta+rs+1} \\ \theta_{\delta+rs+2} \\ \theta_{\delta+rs+3} \\ \vdots \\ \theta_{\delta+rs+r} \end{pmatrix} \tag{5}$$
$$= \Big(\Big[\textstyle\sum_{k=1}^{s} x_k \theta_{\delta+k} \Big] + \theta_{\delta+rs+1}, \Big[\textstyle\sum_{k=1}^{s} x_k \theta_{\delta+s+k} \Big] + \theta_{\delta+rs+2}, \ldots, \Big[\textstyle\sum_{k=1}^{s} x_k \theta_{\delta+(r-1)s+k} \Big] + \theta_{\delta+rs+r} \Big).$$

2.1.2 Vectorized description of DNNs

Definition 2.2. Let $d, L \in \mathbb{N}$, $l_0, l_1, \ldots, l_L \in \mathbb{N}$, $\delta \in \mathbb{N}_0$, $\theta \in \mathbb{R}^d$ satisfy

$$d \geq \delta + \sum_{k=1}^{L} l_k(l_{k-1} + 1) \tag{6}$$

and let $\Psi_k \colon \mathbb{R}^{l_k} \to \mathbb{R}^{l_k}$, $k \in \{1, 2, \ldots, L\}$, be functions. Then we denote by $\mathcal{N}^{\theta,\delta,l_0}_{\Psi_1,\Psi_2,\ldots,\Psi_L} \colon \mathbb{R}^{l_0} \to \mathbb{R}^{l_L}$ the function which satisfies for all $x \in \mathbb{R}^{l_0}$ that

$$\big(\mathcal{N}^{\theta,\delta,l_0}_{\Psi_1,\Psi_2,\ldots,\Psi_L}\big)(x) = \big(\Psi_L \circ \mathcal{A}^{\theta,\delta+\sum_{k=1}^{L-1} l_k(l_{k-1}+1)}_{l_L,l_{L-1}} \circ \Psi_{L-1} \circ \mathcal{A}^{\theta,\delta+\sum_{k=1}^{L-2} l_k(l_{k-1}+1)}_{l_{L-1},l_{L-2}} \circ \ldots$$
$$\ldots \circ \Psi_2 \circ \mathcal{A}^{\theta,\delta+l_1(l_0+1)}_{l_2,l_1} \circ \Psi_1 \circ \mathcal{A}^{\theta,\delta}_{l_1,l_0}\big)(x) \tag{7}$$

(cf. Definition 2.1).

2.1.3 Activation functions

Definition 2.3 (Multidimensional version)**.** Let $d \in \mathbb{N}$ and let $\psi \colon \mathbb{R} \to \mathbb{R}$ be a function. Then we denote by $\mathfrak{M}_{\psi,d} \colon \mathbb{R}^d \to \mathbb{R}^d$ the function which satisfies for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that

$$\mathfrak{M}_{\psi,d}(x) = (\psi(x_1), \psi(x_2), \ldots, \psi(x_d)). \tag{8}$$

Definition 2.4 (Rectifier function)**.** We denote by $\mathfrak{r} \colon \mathbb{R} \to \mathbb{R}$ the function which satisfies for all $x \in \mathbb{R}$ that

$$\mathfrak{r}(x) = \max\{x, 0\}. \tag{9}$$

Definition 2.5 (Multidimensional rectifier function)**.** Let $d \in \mathbb{N}$. Then we denote by $\mathfrak{R}_d \colon \mathbb{R}^d \to \mathbb{R}^d$ the function given by

$$\mathfrak{R}_d = \mathfrak{M}_{\mathfrak{r},d} \tag{10}$$

(cf. Definitions 2.3 and 2.4).

Definition 2.6 (Clipping function)**.** Let $u \in [-\infty, \infty)$, $v \in (u, \infty]$. Then we denote by $\mathfrak{c}_{u,v} \colon \mathbb{R} \to \mathbb{R}$ the function which satisfies for all $x \in \mathbb{R}$ that

$$\mathfrak{c}_{u,v}(x) = \max\{u, \min\{x, v\}\}. \tag{11}$$

Definition 2.7 (Multidimensional clipping function)**.** Let $d \in \mathbb{N}$, $u \in [-\infty, \infty)$, $v \in (u, \infty]$. Then we denote by $\mathfrak{C}_{u,v,d} \colon \mathbb{R}^d \to \mathbb{R}^d$ the function given by

$$\mathfrak{C}_{u,v,d} = \mathfrak{M}_{\mathfrak{c}_{u,v},d} \tag{12}$$

(cf. Definitions 2.3 and 2.6).

2.1.4 Rectified DNNs

Definition 2.8 (Rectified clipped DNN)**.** Let $L, d \in \mathbb{N}$, $u \in [-\infty, \infty)$, $v \in (u, \infty]$, $\mathbf{l} = (l_0, l_1, \dots, l_L) \in \mathbb{N}^{L+1}$, $\theta \in \mathbb{R}^d$ satisfy

$$d \geq \sum_{k=1}^{L} l_k(l_{k-1} + 1). \tag{13}$$

Then we denote by $\mathscr{N}_{u,v}^{\theta,\mathbf{l}} \colon \mathbb{R}^{l_0} \to \mathbb{R}^{l_L}$ the function which satisfies for all $x \in \mathbb{R}^{l_0}$ that

$$\mathscr{N}_{u,v}^{\theta,\mathbf{l}}(x) = \begin{cases} \big(\mathcal{N}^{\theta,0,l_0}_{\mathfrak{C}_{u,v,l_L}}\big)(x) & : L = 1 \\ \big(\mathcal{N}^{\theta,0,l_0}_{\mathfrak{R}_{l_1},\mathfrak{R}_{l_2},\dots,\mathfrak{R}_{l_{L-1}},\mathfrak{C}_{u,v,l_L}}\big)(x) & : L > 1 \end{cases} \tag{14}$$

(cf. Definitions 2.7, 2.5, and 2.2).

2.2 Structured description of DNNs

2.2.1 Structured description of DNNs

Definition 2.9. We denote by $\mathbf{N}$ the set given by

$$\mathbf{N} = \textstyle\bigcup_{L \in \mathbb{N}} \bigcup_{(l_0,l_1,\dots,l_L) \in \mathbb{N}^{L+1}} \big(\bigtimes_{k=1}^{L} (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}) \big) \tag{15}$$

and we denote by $\mathcal{P}, \mathcal{L}, \mathcal{I}, \mathcal{O} \colon \mathbf{N} \to \mathbb{N}$, $\mathcal{H} \colon \mathbf{N} \to \mathbb{N}_0$, and $\mathcal{D} \colon \mathbf{N} \to \big(\bigcup_{L=2}^{\infty} \mathbb{N}^L\big)$ the functions which satisfy for all $L \in \mathbb{N}$, $l_0, l_1, \dots, l_L \in \mathbb{N}$, $\Phi \in \big(\bigtimes_{k=1}^{L} (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k})\big)$ that $\mathcal{P}(\Phi) = \sum_{k=1}^{L} l_k(l_{k-1} + 1)$, $\mathcal{L}(\Phi) = L$, $\mathcal{I}(\Phi) = l_0$, $\mathcal{O}(\Phi) = l_L$, $\mathcal{H}(\Phi) = L - 1$, and $\mathcal{D}(\Phi) = (l_0, l_1, \dots, l_L)$.

2.2.2 Realizations of DNNs

Definition 2.10 (Realization associated to a DNN)**.** Let $a \in C(\mathbb{R}, \mathbb{R})$. Then we denote by $\mathcal{R}_a \colon \mathbf{N} \to \big(\bigcup_{k,l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l)\big)$ the function which satisfies for all $L \in \mathbb{N}$, $l_0, l_1, \dots, l_L \in \mathbb{N}$, $\Phi = ((W_1, B_1), (W_2, B_2), \dots, (W_L, B_L)) \in \big(\bigtimes_{k=1}^{L} (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k})\big)$, $x_0 \in \mathbb{R}^{l_0}, x_1 \in \mathbb{R}^{l_1}, \dots, x_{L-1} \in \mathbb{R}^{l_{L-1}}$ with $\forall\, k \in \mathbb{N} \cap (0, L) \colon x_k = \mathfrak{M}_{a,l_k}(W_k x_{k-1} + B_k)$ that

$$\mathcal{R}_a(\Phi) \in C(\mathbb{R}^{l_0}, \mathbb{R}^{l_L}) \qquad \text{and} \qquad (\mathcal{R}_a(\Phi))(x_0) = W_L x_{L-1} + B_L \tag{16}$$

(cf. Definitions 2.9 and 2.3).

2.2.3 On the connection to the vectorized description of DNNs

Definition 2.11. We denote by $\mathcal{T} \colon \mathbf{N} \to \big(\bigcup_{d \in \mathbb{N}} \mathbb{R}^d\big)$ the function which satisfies for all $L, d \in \mathbb{N}$, $l_0, l_1, \dots, l_L \in \mathbb{N}$, $\Phi = ((W_1, B_1), (W_2, B_2), \dots, (W_L, B_L)) \in \big(\bigtimes_{m=1}^{L} (\mathbb{R}^{l_m \times l_{m-1}} \times \mathbb{R}^{l_m})\big)$, $\theta = (\theta_1, \theta_2, \dots, \theta_d) \in$

$\mathbb{R}^d$, $k \in \{1, 2, \ldots, L\}$ with $\mathcal{T}(\Phi) = \theta$ that

$$d = \mathcal{P}(\Phi), \qquad B_k = \begin{pmatrix} \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+l_k l_{k-1}+1} \\ \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+l_k l_{k-1}+2} \\ \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+l_k l_{k-1}+3} \\ \vdots \\ \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+l_k l_{k-1}+l_k} \end{pmatrix}, \qquad \text{and}$$

$$W_k = \begin{pmatrix} \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+1} & \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+2} & \cdots & \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+l_{k-1}} \\ \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+l_{k-1}+1} & \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+l_{k-1}+2} & \cdots & \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+2l_{k-1}} \\ \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+2l_{k-1}+1} & \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+2l_{k-1}+2} & \cdots & \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+3l_{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+(l_k-1)l_{k-1}+1} & \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+(l_k-1)l_{k-1}+2} & \cdots & \theta_{(\sum_{i=1}^{k-1} l_i(l_{i-1}+1))+l_k l_{k-1}} \end{pmatrix}, \tag{17}$$

(cf. Definition 2.9).

Lemma 2.12. *Let $a, b \in \mathbb{N}$, $W = (W_{i,j})_{(i,j)\in\{1,2,\ldots,a\}\times\{1,2,\ldots,b\}} \in \mathbb{R}^{a\times b}$, $B = (B_i)_{i\in\{1,2,\ldots,a\}} \in \mathbb{R}^a$. Then*

$$\mathcal{T}\big(((W, B))\big) = \big(W_{1,1}, W_{1,2}, \ldots, W_{1,b}, W_{2,1}, W_{2,2}, \ldots, W_{2,b}, \ldots, W_{a,1}, W_{a,2}, \ldots, W_{a,b}, B_1, B_2, \ldots, B_a\big) \tag{18}$$

(cf. Definition 2.11).

Proof of Lemma 2.12. Observe that (17) establishes (18). The proof of Lemma 2.12 is thus completed. □

Lemma 2.13. *Let $L \in \mathbb{N}$, $l_0, l_1, \ldots, l_L \in \mathbb{N}$, let $W_k = (W_{k,i,j})_{(i,j)\in\{1,2,\ldots,l_k\}\times\{1,2,\ldots,l_{k-1}\}} \in \mathbb{R}^{l_k\times l_{k-1}}$, $k \in \{1, 2, \ldots, L\}$, and let $B_k = (B_{k,i})_{i\in\{1,2,\ldots,l_k\}} \in \mathbb{R}^{l_k}$, $k \in \{1, 2, \ldots, L\}$. Then*

(i) it holds for all $k \in \{1, 2, \ldots, L\}$ that

$$\mathcal{T}\big(((W_k, B_k))\big) = \big(W_{k,1,1}, W_{k,1,2}, \ldots, W_{k,1,l_{k-1}}, W_{k,2,1}, W_{k,2,2}, \ldots, W_{k,2,l_{k-1}}, \ldots, \\ W_{k,l_k,1}, W_{k,l_k,2}, \ldots, W_{k,l_k,l_{k-1}}, B_{k,1}, B_{k,2}, \ldots, B_{k,l_k}\big) \tag{19}$$

and

(ii) it holds that

$$\begin{aligned} &\mathcal{T}\Big(\big((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L)\big)\Big) \\ &= \Big(W_{1,1,1}, W_{1,1,2}, \ldots, W_{1,1,l_0}, \ldots, W_{1,l_1,1}, W_{1,l_1,2}, \ldots, W_{1,l_1,l_0}, B_{1,1}, B_{1,2}, \ldots, B_{1,l_1}, \\ &\quad W_{2,1,1}, W_{2,1,2}, \ldots, W_{2,1,l_1}, \ldots, W_{2,l_2,1}, W_{2,l_2,2}, \ldots, W_{2,l_2,l_1}, B_{2,1}, B_{2,2}, \ldots, B_{2,l_2}, \\ &\quad \ldots, \\ &\quad W_{L,1,1}, W_{L,1,2}, \ldots, W_{L,1,l_{L-1}}, \ldots W_{L,l_L,1}, W_{L,l_L,2}, \ldots, W_{L,l_L,l_{L-1}}, B_{L,1}, B_{L,2}, \ldots, B_{L,l_L}\Big) \end{aligned} \tag{20}$$

(cf. Definition 2.11).

Proof of Lemma 2.13. Note that Lemma 2.12 proves item (i). Moreover, observe that (17) establishes item (ii). The proof of Lemma 2.13 is thus completed. □

Lemma 2.14. *Let $a \in C(\mathbb{R}, \mathbb{R})$, $\Phi \in \mathbf{N}$, $L \in \mathbb{N}$, $l_0, l_1, \ldots, l_L \in \mathbb{N}$ satisfy $\mathcal{D}(\Phi) = (l_0, l_1, \ldots, l_L)$ (cf. Definition 2.9). Then it holds for all $x \in \mathbb{R}^{l_0}$ that*

$$(\mathcal{R}_a(\Phi))(x) = \begin{cases} \big(\mathcal{N}^{\mathcal{T}(\Phi),0,l_0}_{\mathrm{id}_{\mathbb{R}^{l_L}}}\big)(x) & : L = 1 \\ \big(\mathcal{N}^{\mathcal{T}(\Phi),0,l_0}_{\mathfrak{M}_{a,l_1},\mathfrak{M}_{a,l_2},\ldots,\mathfrak{M}_{a,l_{L-1}},\mathrm{id}_{\mathbb{R}^{l_L}}}\big)(x) & : L > 1 \end{cases} \tag{21}$$

(cf. Definitions 2.10, 2.11, 2.3, and 2.2).

Proof of Lemma 2.14. Throughout this proof let $W_1 \in \mathbb{R}^{l_1 \times l_0}$, $B_1 \in \mathbb{R}^{l_1}$, $W_2 \in \mathbb{R}^{l_2 \times l_1}$, $B_2 \in \mathbb{R}^{l_2}$, $\ldots$, $W_L \in \mathbb{R}^{l_L \times l_{L-1}}$, $B_L \in \mathbb{R}^{l_L}$ satisfy $\Phi = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L))$. Note that (17) shows that for all $k \in \{1, 2, \ldots, L\}$, $x \in \mathbb{R}^{l_{k-1}}$ it holds that

$$W_k x + B_k = \big(\mathcal{A}_{l_k, l_{k-1}}^{\mathcal{T}(\Phi), \sum_{i=1}^{k-1} l_i(l_{i-1}+1)}\big)(x) \tag{22}$$

(cf. Definitions 2.11 and 2.1). This demonstrates that for all $x_0 \in \mathbb{R}^{l_0}$, $x_1 \in \mathbb{R}^{l_1}, \ldots, x_{L-1} \in \mathbb{R}^{l_{L-1}}$ with $\forall\, k \in \mathbb{N} \cap (0, L)\colon x_k = \mathfrak{M}_{a, l_k}(W_k x_{k-1} + B_k)$ it holds that

$$x_{L-1} = \begin{cases} x_0 & : L = 1 \\ \big(\mathfrak{M}_{a, l_{L-1}} \circ \mathcal{A}_{l_{L-1}, l_{L-2}}^{\mathcal{T}(\Phi), \sum_{i=1}^{L-2} l_i(l_{i-1}+1)} \circ \mathfrak{M}_{a, l_{L-2}} \circ \mathcal{A}_{l_{L-2}, l_{L-3}}^{\mathcal{T}(\Phi), \sum_{i=1}^{L-3} l_i(l_{i-1}+1)} \circ \ldots \circ \mathfrak{M}_{a, l_1} \circ \mathcal{A}_{l_1, l_0}^{\mathcal{T}(\Phi), 0}\big)(x_0) & : L > 1 \end{cases} \tag{23}$$

(cf. Definition 2.3). Combining this and (22) with (7) and (16) proves that for all $x_0 \in \mathbb{R}^{l_0}$, $x_1 \in \mathbb{R}^{l_1}, \ldots, x_{L-1} \in \mathbb{R}^{l_{L-1}}$ with $\forall\, k \in \mathbb{N} \cap (0, L)\colon x_k = \mathfrak{M}_{a, l_k}(W_k x_{k-1} + B_k)$ it holds that

$$\begin{aligned} \big(\mathcal{R}_a(\Phi)\big)(x_0) &= W_L x_{L-1} + B_L = \big(\mathcal{A}_{l_L, l_{L-1}}^{\mathcal{T}(\Phi), \sum_{i=1}^{L-1} l_i(l_{i-1}+1)}\big)(x_{L-1}) \\ &= \begin{cases} \big(\mathcal{N}_{\mathrm{id}_{\mathbb{R}^{l_L}}}^{\mathcal{T}(\Phi), 0, l_0}\big)(x_0) & : L = 1 \\ \big(\mathcal{N}_{\mathfrak{M}_{a, l_1}, \mathfrak{M}_{a, l_2}, \ldots, \mathfrak{M}_{a, l_{L-1}}, \mathrm{id}_{\mathbb{R}^{l_L}}}^{\mathcal{T}(\Phi), 0, l_0}\big)(x_0) & : L > 1 \end{cases} \end{aligned} \tag{24}$$

(cf. Definitions 2.10 and 2.2). The proof of Lemma 2.14 is thus completed. □

Corollary 2.15. *Let $\Phi \in \mathbf{N}$ (cf. Definition 2.9). Then it holds for all $x \in \mathbb{R}^{\mathcal{I}(\Phi)}$ that*

$$\big(\mathscr{N}_{-\infty, \infty}^{\mathcal{T}(\Phi), \mathcal{D}(\Phi)}\big)(x) = (\mathcal{R}_{\mathfrak{r}}(\Phi))(x) \tag{25}$$

(cf. Definitions 2.11, 2.8, 2.4, and 2.10).

Proof of Corollary 2.15. Note that Lemma 2.14, (14), (10), and the fact that for all $d \in \mathbb{N}$ it holds that $\mathfrak{C}_{-\infty, \infty, d} = \mathrm{id}_{\mathbb{R}^d}$ establish (25) (cf. Definition 2.7). The proof of Corollary 2.15 is thus completed. □

2.2.4 Parallelizations of DNNs

Definition 2.16 (Parallelization of DNNs)**.** Let $n \in \mathbb{N}$. Then we denote by

$$\mathbf{P}_n\colon \big\{(\Phi_1, \Phi_2, \ldots, \Phi_n) \in \mathbf{N}^n \colon \mathcal{L}(\Phi_1) = \mathcal{L}(\Phi_2) = \ldots = \mathcal{L}(\Phi_n)\big\} \to \mathbf{N} \tag{26}$$

the function which satisfies for all $L \in \mathbb{N}$, $(l_{1,0}, l_{1,1}, \ldots, l_{1,L}), (l_{2,0}, l_{2,1}, \ldots, l_{2,L}), \ldots, (l_{n,0}, l_{n,1}, \ldots, l_{n,L}) \in \mathbb{N}^{L+1}$, $\Phi_1 = ((W_{1,1}, B_{1,1}), (W_{1,2}, B_{1,2}), \ldots, (W_{1,L}, B_{1,L})) \in \big(\bigtimes_{k=1}^{L}(\mathbb{R}^{l_{1,k}\times l_{1,k-1}} \times \mathbb{R}^{l_{1,k}})\big)$, $\Phi_2 = ((W_{2,1}, B_{2,1}), (W_{2,2}, B_{2,2}), \ldots, (W_{2,L}, B_{2,L})) \in \big(\bigtimes_{k=1}^{L}(\mathbb{R}^{l_{2,k}\times l_{2,k-1}} \times \mathbb{R}^{l_{2,k}})\big)$, $\ldots$, $\Phi_n = ((W_{n,1}, B_{n,1}), (W_{n,2}, B_{n,2}), \ldots, (W_{n,L}, B_{n,L})) \in \big(\bigtimes_{k=1}^{L}(\mathbb{R}^{l_{n,k}\times l_{n,k-1}} \times \mathbb{R}^{l_{n,k}})\big)$ that

$$\begin{aligned}\mathbf{P}_n(\Phi_1, \Phi_2, \ldots, \Phi_n) = \Bigg(&\left(\begin{pmatrix} W_{1,1} & 0 & 0 & \cdots & 0 \\ 0 & W_{2,1} & 0 & \cdots & 0 \\ 0 & 0 & W_{3,1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & W_{n,1} \end{pmatrix}, \begin{pmatrix} B_{1,1} \\ B_{2,1} \\ B_{3,1} \\ \vdots \\ B_{n,1} \end{pmatrix}\right), \\ &\left(\begin{pmatrix} W_{1,2} & 0 & 0 & \cdots & 0 \\ 0 & W_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & W_{3,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & W_{n,2} \end{pmatrix}, \begin{pmatrix} B_{1,2} \\ B_{2,2} \\ B_{3,2} \\ \vdots \\ B_{n,2} \end{pmatrix}\right), \ldots, \\ &\left(\begin{pmatrix} W_{1,L} & 0 & 0 & \cdots & 0 \\ 0 & W_{2,L} & 0 & \cdots & 0 \\ 0 & 0 & W_{3,L} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & W_{n,L} \end{pmatrix}, \begin{pmatrix} B_{1,L} \\ B_{2,L} \\ B_{3,L} \\ \vdots \\ B_{n,L} \end{pmatrix}\right)\Bigg)\end{aligned} \tag{27}$$

(cf. Definition 2.9).

2.2.5 Basic examples for DNNs

Definition 2.17 (Linear transformations as DNNs)**.** Let $m, n \in \mathbb{N}$, $W \in \mathbb{R}^{m\times n}$. Then we denote by $\mathfrak{N}_W \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$ the pair given by $\mathfrak{N}_W = (W, 0)$.

Definition 2.18. We denote by $\mathfrak{I} = (\mathfrak{I}_d)_{d\in\mathbb{N}} \colon \mathbb{N} \to \mathbf{N}$ the function which satisfies for all $d \in \mathbb{N}$ that

$$\mathfrak{I}_1 = \left(\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right), \left(\begin{pmatrix} 1 & -1 \end{pmatrix}, 0\right)\right) \in \big((\mathbb{R}^{2\times 1} \times \mathbb{R}^2) \times (\mathbb{R}^{1\times 2} \times \mathbb{R}^1)\big) \tag{28}$$

and

$$\mathfrak{I}_d = \mathbf{P}_d(\mathfrak{I}_1, \mathfrak{I}_1, \ldots, \mathfrak{I}_1) \tag{29}$$

(cf. Definitions 2.9 and 2.16).

2.2.6 Compositions of DNNs

Definition 2.19 (Composition of DNNs)**.** We denote by $(\cdot) \bullet (\cdot)\colon \{(\Phi_1, \Phi_2) \in \mathbf{N} \times \mathbf{N} \colon \mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2)\} \to \mathbf{N}$ the function which satisfies for all $L, \mathfrak{L} \in \mathbb{N}$, $l_0, l_1, \ldots, l_L, \mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_{\mathfrak{L}} \in \mathbb{N}$, $\Phi_1 = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L)) \in \big(\bigtimes_{k=1}^{L} (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k})\big)$, $\Phi_2 = ((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_{\mathfrak{L}}, \mathfrak{B}_{\mathfrak{L}})) \in \big(\bigtimes_{k=1}^{\mathfrak{L}} (\mathbb{R}^{\mathfrak{l}_k \times \mathfrak{l}_{k-1}} \times \mathbb{R}^{\mathfrak{l}_k})\big)$ with $l_0 = \mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2) = \mathfrak{l}_{\mathfrak{L}}$ that

$$\Phi_1 \bullet \Phi_2 = \begin{cases} \begin{array}{r}\big((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_{\mathfrak{L}-1}, \mathfrak{B}_{\mathfrak{L}-1}), (W_1 \mathfrak{W}_{\mathfrak{L}}, W_1 \mathfrak{B}_{\mathfrak{L}} + B_1), \\ (W_2, B_2), (W_3, B_3), \ldots, (W_L, B_L)\big)\end{array} & : L > 1 < \mathfrak{L} \\ \big((W_1 \mathfrak{W}_1, W_1 \mathfrak{B}_1 + B_1), (W_2, B_2), (W_3, B_3), \ldots, (W_L, B_L)\big) & : L > 1 = \mathfrak{L} \\ \big((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_{\mathfrak{L}-1}, \mathfrak{B}_{\mathfrak{L}-1}), (W_1 \mathfrak{W}_{\mathfrak{L}}, W_1 \mathfrak{B}_{\mathfrak{L}} + B_1)\big) & : L = 1 < \mathfrak{L} \\ \big((W_1 \mathfrak{W}_1, W_1 \mathfrak{B}_1 + B_1)\big) & : L = 1 = \mathfrak{L} \end{cases} \tag{30}$$

(cf. Definition 2.9).

Definition 2.20 (Maximum norm)**.** We denote by $|||\cdot|||\colon \big(\bigcup_{d=1}^{\infty} \mathbb{R}^d\big) \to [0, \infty)$ the function which satisfies for all $d \in \mathbb{N}$, $\theta = (\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d$ that

$$|||\theta||| = \max_{i \in \{1,2,\ldots,d\}} |\theta_i|. \tag{31}$$

Lemma 2.21. *Let* $L, \mathfrak{L} \in \mathbb{N}$, $l_0, l_1, \ldots, l_L, \mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_{\mathfrak{L}} \in \mathbb{N}$, $\Phi_1 = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L)) \in \big(\bigtimes_{k=1}^{L} (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k})\big)$, $\Phi_2 = ((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_{\mathfrak{L}}, \mathfrak{B}_{\mathfrak{L}})) \in \big(\bigtimes_{k=1}^{\mathfrak{L}} (\mathbb{R}^{\mathfrak{l}_k \times \mathfrak{l}_{k-1}} \times \mathbb{R}^{\mathfrak{l}_k})\big)$. *Then*

$$|||\mathcal{T}(\Phi_1 \bullet \Phi_2)||| \le \max\big\{|||\mathcal{T}(\Phi_1)|||, |||\mathcal{T}(\Phi_2)|||, \big|\big|\big|\mathcal{T}\big(((W_1 \mathfrak{W}_{\mathfrak{L}}, W_1 \mathfrak{B}_{\mathfrak{L}} + B_1))\big)\big|\big|\big|\big\} \tag{32}$$

(cf. Definitions 2.19, 2.11, and 2.20).

Proof of Lemma 2.21. Note that (30) and Lemma 2.13 establish (32). The proof of Lemma 2.21 is thus completed. ∎

2.2.7 Powers and extensions of DNNs

Definition 2.22. Let $d \in \mathbb{N}$. Then we denote by $\mathrm{I}_d \in \mathbb{R}^{d \times d}$ the identity matrix in $\mathbb{R}^{d \times d}$.

Definition 2.23. We denote by $(\cdot)^{\bullet n}\colon \{\Phi \in \mathbf{N} \colon \mathcal{I}(\Phi) = \mathcal{O}(\Phi)\} \to \mathbf{N}$, $n \in \mathbb{N}_0$, the functions which satisfy for all $n \in \mathbb{N}_0$, $\Phi \in \mathbf{N}$ with $\mathcal{I}(\Phi) = \mathcal{O}(\Phi)$ that

$$\Phi^{\bullet n} = \begin{cases} \big(\mathrm{I}_{\mathcal{O}(\Phi)}, (0, 0, \ldots, 0)\big) \in \mathbb{R}^{\mathcal{O}(\Phi) \times \mathcal{O}(\Phi)} \times \mathbb{R}^{\mathcal{O}(\Phi)} & : n = 0 \\ \Phi \bullet (\Phi^{\bullet (n-1)}) & : n \in \mathbb{N} \end{cases} \tag{33}$$

(cf. Definitions 2.9, 2.22, and 2.19).

Definition 2.24 (Extension of DNNs)**.** Let $L \in \mathbb{N}$, $\Psi \in \mathbf{N}$ satisfy $\mathcal{I}(\Psi) = \mathcal{O}(\Psi)$. Then we denote by $\mathcal{E}_{L,\Psi}\colon \{\Phi \in \mathbf{N} \colon (\mathcal{L}(\Phi) \le L \text{ and } \mathcal{O}(\Phi) = \mathcal{I}(\Psi))\} \to \mathbf{N}$ the function which satisfies for all $\Phi \in \mathbf{N}$ with $\mathcal{L}(\Phi) \le L$ and $\mathcal{O}(\Phi) = \mathcal{I}(\Psi)$ that

$$\mathcal{E}_{L,\Psi}(\Phi) = \big(\Psi^{\bullet (L - \mathcal{L}(\Phi))}\big) \bullet \Phi \tag{34}$$

(cf. Definitions 2.9, 2.23, and 2.19).

Lemma 2.25. *Let $d, \mathfrak{i}, L, \mathfrak{L} \in \mathbb{N}$, $l_0, l_1, \ldots, l_{L-1} \in \mathbb{N}$, $\Phi, \Psi \in \mathbf{N}$ satisfy $\mathfrak{L} \geq L$, $\mathcal{D}(\Phi) = (l_0, l_1, \ldots, l_{L-1}, d)$ and $\mathcal{D}(\Psi) = (d, \mathfrak{i}, d)$ (cf. Definition 2.9). Then it holds that $\mathcal{D}(\mathcal{E}_{\mathfrak{L},\Psi}(\Phi)) \in \mathbb{N}^{\mathfrak{L}+1}$ and*

$$\mathcal{D}(\mathcal{E}_{\mathfrak{L},\Psi}(\Phi)) = \begin{cases} (l_0, l_1, \ldots, l_{L-1}, d) & : \mathfrak{L} = L \\ (l_0, l_1, \ldots, l_{L-1}, \mathfrak{i}, \mathfrak{i}, \ldots, \mathfrak{i}, d) & : \mathfrak{L} > L \end{cases} \tag{35}$$

(cf. Definition 2.24).

Proof of Lemma 2.25. Observe that item (i) in [31, Lemma 2.13] ensures that $\mathcal{L}(\Psi^{\bullet(\mathfrak{L}-L)}) = \mathfrak{L} - L + 1$, $\mathcal{D}(\Psi^{\bullet(\mathfrak{L}-L)}) \in \mathbb{N}^{\mathfrak{L}-L+2}$, and

$$\mathcal{D}(\Psi^{\bullet(\mathfrak{L}-L)}) = \begin{cases} (d, d) & : \mathfrak{L} = L \\ (d, \mathfrak{i}, \mathfrak{i}, \ldots, \mathfrak{i}, d) & : \mathfrak{L} > L \end{cases} \tag{36}$$

(cf. Definition 2.23). Combining this with [31, Proposition 2.6] shows that $\mathcal{L}((\Psi^{\bullet(\mathfrak{L}-L)}) \bullet \Phi) = \mathcal{L}(\Psi^{\bullet(\mathfrak{L}-L)}) + \mathcal{L}(\Phi) - 1 = \mathfrak{L}$, $\mathcal{D}((\Psi^{\bullet(\mathfrak{L}-L)}) \bullet \Phi) \in \mathbb{N}^{\mathfrak{L}+1}$, and

$$\mathcal{D}((\Psi^{\bullet(\mathfrak{L}-L)}) \bullet \Phi) = \begin{cases} (l_0, l_1, \ldots, l_{L-1}, d) & : \mathfrak{L} = L \\ (l_0, l_1, \ldots, l_{L-1}, \mathfrak{i}, \mathfrak{i}, \ldots, \mathfrak{i}, d) & : \mathfrak{L} > L. \end{cases} \tag{37}$$

This and (34) establish (35). The proof of Lemma 2.25 is thus completed. □

Lemma 2.26. *Let $d, L \in \mathbb{N}$, $\Phi \in \mathbf{N}$ satisfy $L \geq \mathcal{L}(\Phi)$ and $d = \mathcal{O}(\Phi)$ (cf. Definition 2.9). Then*

$$|||\mathcal{T}(\mathcal{E}_{L,\mathfrak{I}_d}(\Phi))||| \leq \max\{1, |||\mathcal{T}(\Phi)|||\} \tag{38}$$

(cf. Definitions 2.18, 2.24, 2.11, and 2.20).

Proof of Lemma 2.26. Throughout this proof assume w.l.o.g. that $L > \mathcal{L}(\Phi)$ and let $l_0, l_1, \ldots, l_{L-\mathcal{L}(\Phi)+1} \in \mathbb{N}$ satisfy $(l_0, l_1, \ldots, l_{L-\mathcal{L}(\Phi)+1}) = (d, 2d, 2d, \ldots, 2d, d)$. Note that [32, Lemma 3.16] ensures that $\mathcal{D}(\mathfrak{I}_d) = (d, 2d, d) \in \mathbb{N}^3$ (cf. Definition 2.18). Item (i) in [31, Lemma 2.13] hence establishes that

$$\mathcal{L}((\mathfrak{I}_d)^{\bullet(L-\mathcal{L}(\Phi)}) = L - \mathcal{L}(\Phi) + 1 \qquad \text{and} \qquad \mathcal{D}((\mathfrak{I}_d)^{\bullet(L-\mathcal{L}(\Phi)}) = (l_0, l_1, \ldots, l_{L-\mathcal{L}(\Phi)+1}) \in \mathbb{N}^{L-\mathcal{L}(\Phi)+2} \tag{39}$$

(cf. Definition 2.23). This shows that there exist $W_k \in \mathbb{R}^{l_k \times l_{k-1}}$, $k \in \{1, 2, \ldots, L-\mathcal{L}(\Phi)+1\}$, and $B_k \in \mathbb{R}^{l_k}$, $k \in \{1, 2, \ldots, L - \mathcal{L}(\Phi) + 1\}$, which satisfy

$$(\mathfrak{I}_d)^{\bullet(L-\mathcal{L}(\Phi))} = ((W_1, B_1), (W_2, B_2), \ldots, (W_{L-\mathcal{L}(\Phi)+1}, B_{L-\mathcal{L}(\Phi)+1})). \tag{40}$$

Next observe that (27), (28), (29), (30), and (33) demonstrate that

$$W_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \in \mathbb{R}^{(2d) \times d}$$
$$\text{and} \qquad W_{L-\mathcal{L}(\Phi)+1} = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in \mathbb{R}^{d \times (2d)}. \tag{41}$$

Moreover, note that (27), (28), (29), (30), and (33) prove that for all $k \in \mathbb{N} \cap (1, L - \mathcal{L}(\Phi) + 1)$ it holds that

$$W_k = \underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}}_{\in \mathbb{R}^{d \times (2d)}} \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix}}_{\in \mathbb{R}^{(2d) \times d}}$$

$$= \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(2d) \times (2d)}. \tag{42}$$

In addition, observe that (28), (29), (27), (33), and (30) show that for all $k \in \mathbb{N} \cap [1, L - \mathcal{L}(\Phi)]$ it holds that

$$B_k = 0 \in \mathbb{R}^{2d} \qquad \text{and} \qquad B_{L-\mathcal{L}(\Phi)+1} = 0 \in \mathbb{R}^{d}. \tag{43}$$

Combining this, (41), and (42) establishes that

$$\left|\!\left|\!\left| \mathcal{T}\big((\mathfrak{I}_d)^{\bullet (L-\mathcal{L}(\Phi))}\big) \right|\!\right|\!\right| = 1 \tag{44}$$

(cf. Definitions 2.11 and 2.20). Furthermore, note that (41) demonstrates that for all $k \in \mathbb{N}$, $\mathfrak{W} = (w_{i,j})_{(i,j) \in \{1,2,\ldots,d\} \times \{1,2,\ldots,k\}} \in \mathbb{R}^{d \times k}$ it holds that

$$W_1 \mathfrak{W} = \begin{pmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,k} \\ -w_{1,1} & -w_{1,2} & \cdots & -w_{1,k} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,k} \\ -w_{2,1} & -w_{2,2} & \cdots & -w_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{d,1} & w_{d,2} & \cdots & w_{d,k} \\ -w_{d,1} & -w_{d,2} & \cdots & -w_{d,k} \end{pmatrix} \in \mathbb{R}^{(2d) \times k}. \tag{45}$$

Next observe that (41) and (43) show that for all $\mathfrak{B} = (b_1, b_2, \ldots, b_d) \in \mathbb{R}^d$ it holds that

$$W_1 \mathfrak{B} + B_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{pmatrix} = \begin{pmatrix} b_1 \\ -b_1 \\ b_2 \\ -b_2 \\ \vdots \\ b_d \\ -b_d \end{pmatrix} \in \mathbb{R}^{2d}. \tag{46}$$

Combining this with (45) proves that for all $k \in \mathbb{N}$, $\mathfrak{W} \in \mathbb{R}^{d \times k}$, $\mathfrak{B} \in \mathbb{R}^d$ it holds that

$$\left|\!\left|\!\left| \mathcal{T}\big(((W_1 \mathfrak{W}, W_1 \mathfrak{B} + B_1))\big) \right|\!\right|\!\right| = \left|\!\left|\!\left| \mathcal{T}\big(((\mathfrak{W}, \mathfrak{B}))\big) \right|\!\right|\!\right|. \tag{47}$$

This, Lemma 2.21, and (44) establish that

$$\begin{aligned} |||\mathcal{T}(\mathcal{E}_{L,\mathfrak{I}_d}(\Phi))||| &= \big|\big|\big|\mathcal{T}\big(((\mathfrak{I}_d)^{\bullet(L-\mathcal{L}(\Phi))}) \bullet \Phi\big)\big|\big|\big| \\ &\leq \max\big\{\big|\big|\big|\mathcal{T}\big((\mathfrak{I}_d)^{\bullet(L-\mathcal{L}(\Phi))}\big)\big|\big|\big|, |||\mathcal{T}(\Phi)|||\big\} = \max\{1, |||\mathcal{T}(\Phi)|||\} \end{aligned} \tag{48}$$

(cf. Definition 2.24). The proof of Lemma 2.26 is thus completed. □

2.2.8 Embedding DNNs in larger architectures

Lemma 2.27. *Let $a \in C(\mathbb{R}, \mathbb{R})$, $L \in \mathbb{N}$, $l_0, l_1, \ldots, l_L, \mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_L \in \mathbb{N}$ satisfy for all $k \in \{1, 2, \ldots, L\}$ that $\mathfrak{l}_0 = l_0$, $\mathfrak{l}_L = l_L$, and $\mathfrak{l}_k \geq l_k$, for every $k \in \{1, 2, \ldots, L\}$ let $W_k = (W_{k,i,j})_{(i,j)\in\{1,2,\ldots,l_k\}\times\{1,2,\ldots,l_{k-1}\}} \in \mathbb{R}^{l_k \times l_{k-1}}$, $\mathfrak{W}_k = (\mathfrak{W}_{k,i,j})_{(i,j)\in\{1,2,\ldots,\mathfrak{l}_k\}\times\{1,2,\ldots,\mathfrak{l}_{k-1}\}} \in \mathbb{R}^{\mathfrak{l}_k \times \mathfrak{l}_{k-1}}$, $B_k = (B_{k,i})_{i\in\{1,2,\ldots,l_k\}} \in \mathbb{R}^{l_k}$, $\mathfrak{B}_k = (\mathfrak{B}_{k,i})_{i\in\{1,2,\ldots,\mathfrak{l}_k\}} \in \mathbb{R}^{\mathfrak{l}_k}$, assume for all $k \in \{1, 2, \ldots, L\}$, $i \in \{1, 2, \ldots, l_k\}$, $j \in \mathbb{N} \cap (0, l_{k-1}]$ that $\mathfrak{W}_{k,i,j} = W_{k,i,j}$ and $\mathfrak{B}_{k,i} = B_{k,i}$, and assume for all $k \in \{1, 2, \ldots, L\}$, $i \in \{1, 2, \ldots, l_k\}$, $j \in \mathbb{N} \cap (l_{k-1}, \mathfrak{l}_{k-1} + 1)$ that $\mathfrak{W}_{k,i,j} = 0$. Then*

$$\mathcal{R}_a\big(((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L))\big) = \mathcal{R}_a\big(((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_L, \mathfrak{B}_L))\big) \tag{49}$$

(cf. Definition 2.10).

Proof of Lemma 2.27. Throughout this proof let $\pi_k \colon \mathbb{R}^{\mathfrak{l}_k} \to \mathbb{R}^{l_k}$, $k \in \{0, 1, \ldots, L\}$, satisfy for all $k \in \{0, 1, \ldots, L\}$, $x = (x_1, x_2, \ldots, x_{\mathfrak{l}_k})$ that

$$\pi_k(x) = (x_1, x_2, \ldots, x_{l_k}). \tag{50}$$

Observe that the hypothesis that $\mathfrak{l}_0 = l_0$ and $\mathfrak{l}_L = l_L$ shows that

$$\mathcal{R}_a\big(((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L))\big) \in C(\mathbb{R}^{\mathfrak{l}_0}, \mathbb{R}^{\mathfrak{l}_L}) \tag{51}$$

(cf. Definition 2.10). Furthermore, note that the hypothesis that for all $k \in \{1, 2, \ldots, L\}$, $i \in \{1, 2, \ldots, l_k\}$, $j \in \mathbb{N} \cap (l_{k-1}, \mathfrak{l}_{k-1}+1)$ it holds that $\mathfrak{W}_{k,i,j} = 0$ ensures that for all $k \in \{1, 2, \ldots, L\}$, $x = (x_1, x_2, \ldots, x_{\mathfrak{l}_{k-1}}) \in \mathbb{R}^{\mathfrak{l}_{k-1}}$ it holds that

$$\begin{aligned} \pi_k(\mathfrak{W}_k x + \mathfrak{B}_k) &= \left(\left[\sum_{i=1}^{\mathfrak{l}_{k-1}} \mathfrak{W}_{k,1,i} x_i\right] + \mathfrak{B}_{k,1}, \left[\sum_{i=1}^{\mathfrak{l}_{k-1}} \mathfrak{W}_{k,2,i} x_i\right] + \mathfrak{B}_{k,2}, \ldots, \left[\sum_{i=1}^{\mathfrak{l}_{k-1}} \mathfrak{W}_{k,l_k,i} x_i\right] + \mathfrak{B}_{k,l_k}\right) \\ &= \left(\left[\sum_{i=1}^{l_{k-1}} \mathfrak{W}_{k,1,i} x_i\right] + \mathfrak{B}_{k,1}, \left[\sum_{i=1}^{l_{k-1}} \mathfrak{W}_{k,2,i} x_i\right] + \mathfrak{B}_{k,2}, \ldots, \left[\sum_{i=1}^{l_{k-1}} \mathfrak{W}_{k,l_k,i} x_i\right] + \mathfrak{B}_{k,l_k}\right). \end{aligned} \tag{52}$$

Combining this with the hypothesis that for all $k \in \{1, 2, \ldots, L\}$, $i \in \{1, 2, \ldots, l_k\}$, $j \in \mathbb{N} \cap (0, l_{k-1}]$ it holds that $\mathfrak{W}_{k,i,j} = W_{k,i,j}$ and $\mathfrak{B}_{k,i} = B_{k,i}$ shows that for all $k \in \{1, 2, \ldots, L\}$, $x = (x_1, x_2, \ldots, x_{\mathfrak{l}_{k-1}}) \in \mathbb{R}^{\mathfrak{l}_{k-1}}$ it holds that

$$\begin{aligned} \pi_k(\mathfrak{W}_k x + \mathfrak{B}_k) &= \left(\left[\sum_{i=1}^{l_{k-1}} W_{k,1,i} x_i\right] + B_{k,1}, \left[\sum_{i=1}^{l_{k-1}} W_{k,2,i} x_i\right] + B_{k,2}, \ldots, \left[\sum_{i=1}^{l_{k-1}} W_{k,l_k,i} x_i\right] + B_{k,l_k}\right) \\ &= W_k(\pi_{k-1}(x)) + B_k. \end{aligned} \tag{53}$$

Moreover, observe that (50) and (8) ensure that for all $k \in \{0, 1, \ldots, L\}$, $x = (x_1, x_2, \ldots, x_{\mathfrak{l}_k}) \in \mathbb{R}^{\mathfrak{l}_k}$ it holds that

$$\pi_k(\mathfrak{M}_{a,\mathfrak{l}_k}(x)) = \pi_k(a(x_1), a(x_2), \ldots, a(x_{\mathfrak{l}_k})) = (a(x_1), a(x_2), \ldots, a(x_{l_k})) = \mathfrak{M}_{a,l_k}(\pi_k(x)). \tag{54}$$

Combining this and (53) demonstrates that for all $x_0 \in \mathbb{R}^{\mathfrak{l}_0}$, $x_1 \in \mathbb{R}^{\mathfrak{l}_1}, \ldots, x_{L-1} \in \mathbb{R}^{\mathfrak{l}_{L-1}}$, $k \in \mathbb{N} \cap (0, L)$ with $\forall\, m \in \mathbb{N} \cap (0, L) \colon x_m = \mathfrak{M}_{a,\mathfrak{l}_m}(\mathfrak{W}_m x_{m-1} + \mathfrak{B}_m)$ it holds that

$$\pi_k(x_k) = \pi_k(\mathfrak{M}_{a,\mathfrak{l}_k}(\mathfrak{W}_k x_{k-1} + \mathfrak{B}_k)) = \mathfrak{M}_{a,l_k}(\pi_k(\mathfrak{W}_k x_{k-1} + \mathfrak{B}_k)) = \mathfrak{M}_{a,l_k}(W_k \pi_{k-1}(x_{k-1}) + B_k) \tag{55}$$

(cf. Definition 2.3). The hypothesis that $l_0 = \mathfrak{l}_0$ and $l_L = \mathfrak{l}_L$ and (53) therefore prove that for all $x_0 \in \mathbb{R}^{\mathfrak{l}_0}$, $x_1 \in \mathbb{R}^{\mathfrak{l}_1}, \ldots, x_{L-1} \in \mathbb{R}^{\mathfrak{l}_{L-1}}$ with $\forall\, k \in \mathbb{N} \cap (0, L)\colon x_k = \mathfrak{M}_{a,\mathfrak{l}_k}(\mathfrak{W}_k x_{k-1} + \mathfrak{B}_k)$ it holds that

$$\begin{aligned}\big(\mathcal{R}_a\big(((W_1,B_1),(W_2,B_2),\ldots,(W_L,B_L))\big)\big)(x_0) &= \big(\mathcal{R}_a\big(((W_1,B_1),(W_2,B_2),\ldots,(W_L,B_L))\big)\big)(\pi_0(x_0)) \\ &= W_L \pi_{L-1}(x_{L-1}) + B_L \\ &= \pi_L(\mathfrak{W}_L x_{L-1} + \mathfrak{B}_L) = \mathfrak{W}_L x_{L-1} + \mathfrak{B}_L \\ &= \big(\mathcal{R}_a\big(((\mathfrak{W}_1,\mathfrak{B}_1),(\mathfrak{W}_2,\mathfrak{B}_2),\ldots,(\mathfrak{W}_L,\mathfrak{B}_L))\big)\big)(x_0)\end{aligned} \tag{56}$$

(cf. Definition 2.10). The proof of Lemma 2.27 is thus completed. □

Lemma 2.28. *Let $a \in C(\mathbb{R}, \mathbb{R})$, $L \in \mathbb{N}$, $l_0, l_1, \ldots, l_L, \mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_L \in \mathbb{N}$ satisfy for all $k \in \{1, 2, \ldots, L\}$ that $\mathfrak{l}_0 = l_0$, $\mathfrak{l}_L = l_L$, and $\mathfrak{l}_k \geq l_k$ and let $\Phi \in \mathbf{N}$ satisfy $\mathcal{D}(\Phi) = (l_0, l_1, \ldots, l_L)$ (cf. Definition 2.9). Then there exists $\Psi \in \mathbf{N}$ such that*

$$\mathcal{D}(\Psi) = (\mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_L), \qquad |||\mathcal{T}(\Psi)||| = |||\mathcal{T}(\Phi)|||, \qquad \textit{and} \qquad \mathcal{R}_a(\Psi) = \mathcal{R}_a(\Phi) \tag{57}$$

(cf. Definitions 2.11, 2.20, and 2.10).

Proof of Lemma 2.28. Throughout this proof let $B_k = (B_{k,i})_{i \in \{1,2,\ldots,l_k\}} \in \mathbb{R}^{l_k}$, $k \in \{1, 2, \ldots, L\}$, and $W_k = (W_{k,i,j})_{(i,j) \in \{1,2,\ldots,l_k\} \times \{1,2,\ldots,l_{k-1}\}} \in \mathbb{R}^{l_k \times l_{k-1}}$, $k \in \{1, 2, \ldots, L\}$, satisfy $\Phi = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L))$ and let $\mathfrak{W}_k = (\mathfrak{W}_{k,i,j})_{(i,j) \in \{1,2,\ldots,\mathfrak{l}_k\} \times \{1,2,\ldots,\mathfrak{l}_{k-1}\}} \in \mathbb{R}^{\mathfrak{l}_k \times \mathfrak{l}_{k-1}}$, $k \in \{1, 2, \ldots, L\}$, and $\mathfrak{B}_k = (\mathfrak{B}_{k,i})_{i \in \{1,2,\ldots,\mathfrak{l}_k\}} \in \mathbb{R}^{\mathfrak{l}_k}$, $k \in \{1, 2, \ldots, L\}$, satisfy for all $k \in \{1, 2, \ldots, L\}$, $i \in \{1, 2, \ldots, \mathfrak{l}_k\}$, $j \in \{1, 2, \ldots, \mathfrak{l}_{k-1}\}$ that

$$\mathfrak{W}_{k,i,j} = \begin{cases} W_{k,i,j} & : (i \leq l_k) \wedge (j \leq l_{k-1}) \\ 0 & : (i > l_k) \vee (j > l_{k-1}) \end{cases} \qquad \text{and} \qquad \mathfrak{B}_{k,i} = \begin{cases} B_{k,i} & : i \leq l_k \\ 0 & : i > l_k. \end{cases} \tag{58}$$

Note that (15) ensures that $((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_L, \mathfrak{B}_L)) \in \big(\times_{i=1}^{L} (\mathbb{R}^{\mathfrak{l}_i \times \mathfrak{l}_{i-1}} \times \mathbb{R}^{\mathfrak{l}_i})\big) \subseteq \mathbf{N}$ and

$$\mathcal{D}\big(((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_L, \mathfrak{B}_L))\big) = (\mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_L). \tag{59}$$

Furthermore, observe that Lemma 2.13 and (58) show that

$$|||\mathcal{T}\big(((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_L, \mathfrak{B}_L))\big)||| = |||\mathcal{T}(\Phi)||| \tag{60}$$

(cf. Definitions 2.11 and 2.20). In addition, note that Lemma 2.27 establishes that

$$\mathcal{R}_a(\Phi) = \mathcal{R}_a\big(((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L))\big) = \mathcal{R}_a\big(((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_L, \mathfrak{B}_L))\big) \tag{61}$$

(cf. Definition 2.10). The proof of Lemma 2.28 is thus completed. □

Lemma 2.29. *Let $L, \mathfrak{L} \in \mathbb{N}$, $l_0, l_1, \ldots, l_L, \mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_{\mathfrak{L}} \in \mathbb{N}$ satisfy for all $i \in \mathbb{N} \cap [0, L)$ that $\mathfrak{L} \geq L$ $\mathfrak{l}_0 = l_0$, $\mathfrak{l}_{\mathfrak{L}} = l_L$, and $\mathfrak{l}_i \geq l_i$, assume for all $i \in \mathbb{N} \cap (L-1, \mathfrak{L})$ that $\mathfrak{l}_i \geq 2l_L$, and let $\Phi \in \mathbf{N}$ satisfy $\mathcal{D}(\Phi) = (l_0, l_1, \ldots, l_L)$ (cf. Definition 2.9). Then there exists $\Psi \in \mathbf{N}$ such that*

$$\mathcal{D}(\Psi) = (\mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_{\mathfrak{L}}), \qquad |||\mathcal{T}(\Psi)||| \leq \max\{1, |||\mathcal{T}(\Phi)|||\}, \qquad \textit{and} \qquad \mathcal{R}_{\mathfrak{r}}(\Psi) = \mathcal{R}_{\mathfrak{r}}(\Phi) \tag{62}$$

(cf. Definitions 2.11, 2.20, 2.4, and 2.10).

Proof of Lemma 2.29. Throughout this proof let $\Xi \in \mathbf{N}$ satisfy $\Xi = \mathcal{E}_{\mathfrak{L}, \mathfrak{I}_{l_L}}(\Phi)$ (cf. Definitions 2.18 and 2.24). Note that item (i) in [32, Lemma 3.16] demonstrates that $\mathcal{D}(\mathfrak{I}_{l_L}) = (l_L, 2l_L, l_L) \in \mathbb{N}^3$. Combining this with Lemma 2.25 shows that $\mathcal{D}(\Xi) \in \mathbb{N}^{\mathfrak{L}+1}$ and

$$\mathcal{D}(\Xi) = \begin{cases} (l_0, l_1, \ldots, l_L) & : \mathfrak{L} = L \\ (l_0, l_1, \ldots, l_{L-1}, 2l_L, 2l_L, \ldots, 2l_L, l_L) & : \mathfrak{L} > L. \end{cases} \tag{63}$$

Moreover, observe that Lemma 2.26 (with $d \leftarrow l_L$, $L \leftarrow \mathfrak{L}$, $\Phi \leftarrow \Phi$ in the notation of Lemma 2.26) establishes that

$$|||\mathcal{T}(\Xi)||| \leq \max\{1, |||\mathcal{T}(\Phi)|||\} \tag{64}$$

(cf. Definitions 2.11 and 2.20). In addition, note that item (iii) in [32, Lemma 3.16] ensures that for all $x \in \mathbb{R}^{l_L}$ it holds that

$$(\mathcal{R}_\mathfrak{r}(\mathfrak{I}_{l_L}))(x) = x \tag{65}$$

(cf. Definitions 2.4 and 2.10). This and item (ii) in [31, Lemma 2.14] prove that

$$\mathcal{R}_\mathfrak{r}(\Xi) = \mathcal{R}_\mathfrak{r}(\Phi). \tag{66}$$

In the next step, we observe that (63), the hypothesis that for all $i \in [0, L)$ it holds that $\mathfrak{l}_0 = l_0$, $\mathfrak{l}_\mathfrak{L} = l_L$, and $\mathfrak{l}_i \leq l_i$, the hypothesis that for all $i \in \mathbb{N} \cap (L-1, \mathfrak{L})$ it holds that $\mathfrak{l}_i \geq 2l_L$, and Lemma 2.28 (with $a \leftarrow \mathfrak{r}$, $L \leftarrow \mathfrak{L}$, $(l_0, l_1, \dots, l_L) \leftarrow \mathcal{D}(\Xi)$, $(\mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_\mathfrak{L}) \leftarrow (\mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_\mathfrak{L})$, $\Phi \leftarrow \Xi$ in the notation of Lemma 2.28) ensure that there exists $\Psi \in \mathbf{N}$ such that

$$\mathcal{D}(\Psi) = (\mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_\mathfrak{L}), \qquad |||\mathcal{T}(\Psi)||| = |||\mathcal{T}(\Xi)|||, \qquad \text{and} \qquad \mathcal{R}_\mathfrak{r}(\Psi) = \mathcal{R}_\mathfrak{r}(\Xi). \tag{67}$$

Combining this with (64) and (66) establishes (62). The proof of Lemma 2.29 is thus completed. ☐

Lemma 2.30. *Let $u \in [-\infty, \infty)$, $v \in (u, \infty]$, $L, \mathfrak{L}, d, \mathfrak{d} \in \mathbb{N}$, $\theta \in \mathbb{R}^d$, $l_0, l_1, \dots, l_L, \mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_\mathfrak{L} \in \mathbb{N}$ satisfy for all $i \in \mathbb{N} \cap [0, L)$ that $d \geq \sum_{i=1}^{L} l_i(l_{i-1}+1)$, $\mathfrak{d} \geq \sum_{i=1}^{\mathfrak{L}} \mathfrak{l}_i(\mathfrak{l}_{i-1}+1)$, $\mathfrak{L} \geq L$, $\mathfrak{l}_0 = l_0$, $\mathfrak{l}_\mathfrak{L} = l_L$, and $\mathfrak{l}_i \geq l_i$ and assume for all $i \in \mathbb{N} \cap (L-1, \mathfrak{L})$ that $\mathfrak{l}_i \geq 2l_L$. Then there exists $\vartheta \in \mathbb{R}^\mathfrak{d}$ such that*

$$|||\vartheta||| \leq \max\{1, |||\theta|||\} \qquad \textit{and} \qquad \mathscr{N}_{u,v}^{\vartheta,(\mathfrak{l}_0,\mathfrak{l}_1,\dots,\mathfrak{l}_\mathfrak{L})} = \mathscr{N}_{u,v}^{\theta,(l_0,l_1,\dots,l_L)} \tag{68}$$

(cf. Definitions 2.20 and 2.8).

Proof of Lemma 2.30. Throughout this proof let $\eta_1, \eta_2, \dots, \eta_d \in \mathbb{R}$ satisfy

$$\theta = (\eta_1, \eta_2, \dots, \eta_d) \tag{69}$$

and let $\Phi \in \left(\times_{i=1}^{L} \mathbb{R}^{l_i \times l_{i-1}} \times \mathbb{R}^{l_i}\right)$ satisfy

$$\mathcal{T}(\Phi) = (\eta_1, \eta_2, \dots, \eta_{\mathcal{P}(\Phi)}) \tag{70}$$

(cf. Definition 2.11). Note that Lemma 2.29 establishes that there exists $\Psi \in \mathbf{N}$ which satisfies

$$\mathcal{D}(\Psi) = (\mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_\mathfrak{L}), \qquad |||\mathcal{T}(\Psi)||| \leq \max\{1, |||\mathcal{T}(\Phi)|||\}, \qquad \text{and} \qquad \mathcal{R}_\mathfrak{r}(\Psi) = \mathcal{R}_\mathfrak{r}(\Phi) \tag{71}$$

(cf. Definitions 2.9, 2.20, 2.4, and 2.10). Next let $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_\mathfrak{d}) \in \mathbb{R}^\mathfrak{d}$ satisfy

$$(\vartheta_1, \vartheta_2, \dots, \vartheta_{\mathcal{P}(\Psi)}) = \mathcal{T}(\Psi) \qquad \text{and} \qquad \forall\, i \in \mathbb{N} \cap (\mathcal{P}(\Psi), \mathfrak{d}+1) \colon \vartheta_i = 0. \tag{72}$$

Note that (69), (70), (71), and (72) show that

$$|||\vartheta||| = |||\mathcal{T}(\Psi)||| \leq \max\{1, |||\mathcal{T}(\Phi)|||\} \leq \max\{1, |||\theta|||\}. \tag{73}$$

Next observe that Corollary 2.15 and (70) establish that for all $x \in \mathbb{R}^{l_0}$ it holds that

$$\left(\mathscr{N}_{-\infty,\infty}^{\theta,(l_0,l_1,\dots,l_L)}\right)(x) = \left(\mathscr{N}_{-\infty,\infty}^{\mathcal{T}(\Phi),\mathcal{D}(\Phi)}\right)(x) = (\mathcal{R}_\mathfrak{r}(\Phi))(x). \tag{74}$$

In addition, observe that Corollary 2.15, (71), and (72) prove that for all $x \in \mathbb{R}^{\mathfrak{l}_0}$ it holds that

$$\left(\mathscr{N}_{-\infty,\infty}^{\vartheta,(\mathfrak{l}_0,\mathfrak{l}_1,\dots,\mathfrak{l}_\mathfrak{L})}\right)(x) = \left(\mathscr{N}_{-\infty,\infty}^{\mathcal{T}(\Psi),\mathcal{D}(\Psi)}\right)(x) = (\mathcal{R}_\mathfrak{r}(\Psi))(x). \tag{75}$$

Combining this and (74) with (71) and the hypothesis that $\mathfrak{l}_0 = l_0$ and $\mathfrak{l}_\mathfrak{L} = l_L$ demonstrates that

$$\mathscr{N}_{-\infty,\infty}^{\theta,(l_0,l_1,\dots,l_L)} = \mathscr{N}_{-\infty,\infty}^{\vartheta,(\mathfrak{l}_0,\mathfrak{l}_1,\dots,\mathfrak{l}_\mathfrak{L})}. \tag{76}$$

Hence, we obtain that

$$\mathscr{N}_{u,v}^{\theta,(l_0,l_1,\dots,l_L)} = \mathfrak{C}_{u,v,l_L} \circ \mathscr{N}_{-\infty,\infty}^{\theta,(l_0,l_1,\dots,l_L)} = \mathfrak{C}_{u,v,\mathfrak{l}_\mathfrak{L}} \circ \mathscr{N}_{-\infty,\infty}^{\vartheta,(\mathfrak{l}_0,\mathfrak{l}_1,\dots,\mathfrak{l}_\mathfrak{L})} = \mathscr{N}_{u,v}^{\vartheta,(\mathfrak{l}_0,\mathfrak{l}_1,\dots,\mathfrak{l}_\mathfrak{L})} \tag{77}$$

(cf. Definition 2.7). This and (73) establish (68). The proof of Lemma 2.30 is thus completed. ☐

2.3 Local Lipschitz continuity of the parametrization function

Lemma 2.31. *Let $a, x, y \in \mathbb{R}$. Then*

$$|\max\{x, a\} - \max\{y, a\}| \leq \max\{x, y\} - \min\{x, y\} = |x - y|. \tag{78}$$

Proof of Lemma 2.31. Observe that

$$\begin{aligned}
&|\max\{x, a\} - \max\{y, a\}| = |\max\{\max\{x, y\}, a\} - \max\{\min\{x, y\}, a\}| \\
&= \max\{\max\{x, y\}, a\} - \max\{\min\{x, y\}, a\} \\
&= \max\Big\{\max\{x, y\} - \max\{\min\{x, y\}, a\}, a - \max\{\min\{x, y\}, a\}\Big\} \\
&\leq \max\Big\{\max\{x, y\} - \max\{\min\{x, y\}, a\}, a - a\Big\} \\
&= \max\Big\{\max\{x, y\} - \max\{\min\{x, y\}, a\}, 0\Big\} \leq \max\Big\{\max\{x, y\} - \min\{x, y\}, 0\Big\} \\
&= \max\{x, y\} - \min\{x, y\} = |\max\{x, y\} - \min\{x, y\}| = |x - y|\,.
\end{aligned} \tag{79}$$

The proof of Lemma 2.31 is thus completed. ☐

Corollary 2.32. *Let $a, x, y \in \mathbb{R}$. Then*

$$|\min\{x, a\} - \min\{y, a\}| \leq \max\{x, y\} - \min\{x, y\} = |x - y|. \tag{80}$$

Proof of Corollary 2.32. Note that Lemma 2.31 ensures that

$$\begin{aligned}
|\min\{x, a\} - \min\{y, a\}| &= |-(\min\{x, a\} - \min\{y, a\})| = |\max\{-x, -a\} - \max\{-y, -a\}| \\
&\leq |(-x) - (-y)| = |x - y|\,.
\end{aligned} \tag{81}$$

The proof of Corollary 2.32 is thus completed. ☐

Lemma 2.33. *Let $d \in \mathbb{N}$, $u \in [-\infty, \infty)$, $v \in (u, \infty]$. Then it holds for all $x, y \in \mathbb{R}^d$ that*

$$|||\mathfrak{C}_{u,v,d}(x) - \mathfrak{C}_{u,v,d}(y)||| \leq |||x - y||| \tag{82}$$

(cf. Definitions 2.7 and 2.20).

Proof of Lemma 2.33. Note that Lemma 2.31, Corollary 2.32, and the fact that for all $x \in \mathbb{R}$ it holds that $\max\{-\infty, x\} = x = \min\{x, \infty\}$ show that for all $x, y \in \mathbb{R}$ it holds that

$$|\mathfrak{c}_{u,v}(x) - \mathfrak{c}_{u,v}(y)| = |\max\{u, \min\{x, v\}\} - \max\{u, \min\{y, v\}\}| \leq |\min\{x, v\} - \min\{y, v\}| \leq |x - y| \tag{83}$$

(cf. Definition 2.6). Hence, we obtain that for all $x = (x_1, x_2, \ldots, x_d), y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$ it holds that

$$|||\mathfrak{C}_{u,v,d}(x) - \mathfrak{C}_{u,v,d}(y)||| = \max_{i \in \{1,2,\ldots,d\}} |\mathfrak{c}_{u,v}(x_i) - \mathfrak{c}_{u,v}(y_i)| \leq \max_{i \in \{1,2,\ldots,d\}} |x_i - y_i| = |||x - y||| \tag{84}$$

(cf. Definitions 2.7 and 2.20). The proof of Lemma 2.33 is thus completed. ☐

Lemma 2.34. *Let $d \in \mathbb{N}$. Then it holds for all $x, y \in \mathbb{R}^d$ that*

$$|||\mathfrak{R}_d(x) - \mathfrak{R}_d(y)||| \leq |||x - y||| \tag{85}$$

(cf. Definitions 2.5 and 2.20).

Proof of Lemma 2.34. Note that Lemma 2.33 and the fact that $\mathfrak{R}_d = \mathfrak{C}_{0,\infty,d}$ establish (85). The proof of Lemma 2.34 is thus completed. ☐

Lemma 2.35. *Let $a, b \in \mathbb{N}$, $M = (M_{i,j})_{(i,j)\in\{1,2,\ldots,a\}\times\{1,2,\ldots,b\}} \in \mathbb{R}^{a\times b}$. Then*

$$\sup_{v\in\mathbb{R}^b\setminus\{0\}} \left[\frac{|||Mv|||}{|||v|||}\right] = \max_{i\in\{1,2,\ldots,a\}} \left[\sum_{j=1}^{b} |M_{i,j}|\right] \leq b \left[\max_{i\in\{1,2,\ldots,a\}} \max_{j\in\{1,2,\ldots,b\}} |M_{i,j}|\right] \tag{86}$$

(cf. Definition 2.20).

Proof of Lemma 2.35. Observe that

$$\begin{aligned}
\sup_{v\in\mathbb{R}^b} \left[\frac{|||Mv|||}{|||v|||}\right] &= \sup_{v\in\mathbb{R}^b,\, |||v|||\leq 1} |||Mv||| = \sup_{v=(v_1,v_2,\ldots,v_b)\in[-1,1]^b} |||Mv||| \\
&= \sup_{v=(v_1,v_2,\ldots,v_b)\in[-1,1]^b} \left(\max_{i\in\{1,2,\ldots,a\}} \left|\sum_{j=1}^{b} M_{i,j}v_j\right|\right) \\
&= \max_{i\in\{1,2,\ldots,a\}} \left(\sup_{v=(v_1,v_2,\ldots,v_b)\in[-1,1]^b} \left|\sum_{j=1}^{b} M_{i,j}v_j\right|\right) = \max_{i\in\{1,2,\ldots,a\}} \left(\sum_{j=1}^{b} |M_{i,j}|\right)
\end{aligned} \tag{87}$$

(cf. Definition 2.20). The proof of Lemma 2.35 is thus completed. □

Theorem 2.36. *Let $a \in \mathbb{R}$, $b \in [a, \infty)$, $d, L \in \mathbb{N}$, $l = (l_0, l_1, \ldots, l_L) \in \mathbb{N}^{L+1}$ satisfy*

$$d \geq \sum_{k=1}^{L} l_k(l_{k-1} + 1). \tag{88}$$

Then it holds for all $\theta, \vartheta \in \mathbb{R}^d$ that

$$\begin{aligned}
&\sup_{x\in[a,b]^{l_0}} |||\mathscr{N}^{\theta,l}_{-\infty,\infty}(x) - \mathscr{N}^{\vartheta,l}_{-\infty,\infty}(x)||| \\
&\leq \max\{1, |a|, |b|\}|||\theta - \vartheta||| \left[\prod_{m=0}^{L-1}(l_m + 1)\right] \left[\sum_{n=0}^{L-1} \left(\max\{1, |||\theta|||^n\}\, |||\vartheta|||^{L-1-n}\right)\right] \\
&\leq L \max\{1, |a|, |b|\} \left(\max\{1, |||\theta|||, |||\vartheta|||\}\right)^{L-1} \left[\prod_{m=0}^{L-1}(l_m + 1)\right] |||\theta - \vartheta||| \\
&\leq L \max\{1, |a|, |b|\} \left(|||l||| + 1\right)^{L} \left(\max\{1, |||\theta|||, |||\vartheta|||\}\right)^{L-1} |||\theta - \vartheta|||
\end{aligned} \tag{89}$$

(cf. Definitions 2.8 and 2.20).

Proof of Theorem 2.36. Throughout this proof let $\theta_j = (\theta_{j,1}, \theta_{j,2}, \ldots, \theta_{j,d}) \in \mathbb{R}^d$, $j \in \{1, 2\}$, let $\mathfrak{d} \in \mathbb{N}$ satisfy that

$$\mathfrak{d} = \sum_{k=1}^{L} l_k(l_{k-1} + 1), \tag{90}$$

let $W_{j,k} \in \mathbb{R}^{l_k\times l_{k-1}}$, $k \in \{1, 2, \ldots, L\}$, $j \in \{1, 2\}$, and $B_{j,k} \in \mathbb{R}^{l_k}$, $k \in \{1, 2, \ldots, L\}$, $j \in \{1, 2\}$, satisfy for all $j \in \{1, 2\}$, $k \in \{1, 2, \ldots, L\}$ that

$$\mathcal{T}\big(((W_{j,1}, B_{j,1}), (W_{j,2}, B_{j,2}), \ldots, (W_{j,L}, B_{j,L}))\big) = (\theta_{j,1}, \theta_{j,2}, \ldots, \theta_{j,\mathfrak{d}}), \tag{91}$$

let $\phi_{j,k} \in \mathbf{N}$, $k \in \{1, 2, \ldots, L\}$, $j \in \{1, 2\}$, satisfy for all $j \in \{1, 2\}$, $k \in \{1, 2, \ldots, L\}$ that

$$\phi_{j,k} = \big((W_{j,1}, B_{j,1}), (W_{j,2}, B_{j,2}), \ldots, (W_{j,k}, B_{j,k})\big) \in \left[\times_{i=1}^{k} \left(\mathbb{R}^{l_i\times l_{i-1}} \times \mathbb{R}^{l_i}\right)\right], \tag{92}$$

let $D = [a, b]^{l_0}$, let $\mathfrak{m}_{j,k} \in [0, \infty)$, $j \in \{1, 2\}$, $k \in \{0, 1, \ldots, L\}$, satisfy for all $j \in \{1, 2\}$, $k \in \{0, 1, \ldots, L\}$ that

$$\mathfrak{m}_{j,k} = \begin{cases} \max\{1, |a|, |b|\} & : k = 0 \\ \max\{1, \sup_{x\in D} |||(\mathcal{R}_{\mathfrak{r}}(\phi_{j,k}))(x)|||\} & : k > 0, \end{cases} \tag{93}$$

and let $\mathfrak{e}_k \in [0,\infty)$, $k \in \{0,1,\ldots,L\}$, satisfy for all $k \in \{0,1,\ldots,L\}$ that

$$\mathfrak{e}_k = \begin{cases} 0 & : k = 0 \\ \sup_{x\in D} |||(\mathcal{R}_{\mathfrak{r}}(\phi_{1,k}))(x) - (\mathcal{R}_{\mathfrak{r}}(\phi_{2,k}))(x)||| & : k > 0 \end{cases} \tag{94}$$

(cf. Definitions 2.11, 2.4, 2.10, and 2.20). Note that Lemma 2.35 demonstrates that

$$\begin{aligned} \mathfrak{e}_1 &= \sup_{x\in D} |||(\mathcal{R}_{\mathfrak{r}}(\phi_{1,1}))(x) - (\mathcal{R}_{\mathfrak{r}}(\phi_{2,1}))(x)||| = \sup_{x\in D} |||(W_{1,1}x + B_{1,1}) - (W_{2,1}x + B_{2,1})||| \\ &\le \left[\sup_{x\in D} |||(W_{1,1} - W_{2,1})x|||\right] + |||B_{1,1} - B_{2,1}||| \\ &\le \left[\sup_{v\in\mathbb{R}^{l_0}\setminus\{0\}} \left(\frac{|||(W_{1,1} - W_{2,1})v|||}{|||v|||}\right)\right]\left[\sup_{x\in D} |||x|||\right] + |||B_{1,1} - B_{2,1}||| \\ &\le l_0\, |||\theta_1 - \theta_2|||\max\{|a|,|b|\} + |||B_{1,1} - B_{2,1}||| \le l_0\, |||\theta_1 - \theta_2|||\max\{|a|,|b|\} + |||\theta_1 - \theta_2||| \\ &= |||\theta_1 - \theta_2|||\,(l_0 \max\{|a|,|b|\} + 1) \le \mathfrak{m}_{1,0}\, |||\theta_1 - \theta_2|||\,(l_0 + 1)\,. \end{aligned} \tag{95}$$

Moreover, observe that the triangle inequality assures that for all $k \in \{1,2,\ldots,L\} \cap (1,\infty)$ it holds that

$$\begin{aligned} \mathfrak{e}_k &= \sup_{x\in D} |||(\mathcal{R}_{\mathfrak{r}}(\phi_{1,k}))(x) - (\mathcal{R}_{\mathfrak{r}}(\phi_{2,k}))(x)||| \\ &= \sup_{x\in D} \left|\left|\left| \left[W_{1,k}\Big(\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{1,k-1}))(x)\big)\Big) + B_{1,k}\right] - \left[W_{2,k}\Big(\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{2,k-1}))(x)\big)\Big) + B_{2,k}\right]\right|\right|\right| \\ &\le \left[\sup_{x\in D} \left|\left|\left| W_{1,k}\Big(\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{1,k-1}))(x)\big)\Big) - W_{2,k}\Big(\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{2,k-1}))(x)\big)\Big)\right|\right|\right|\right] + |||\theta_1 - \theta_2|||. \end{aligned} \tag{96}$$

The triangle inequality hence implies that for all $j \in \{1,2\}$, $k \in \{1,2,\ldots,L\} \cap (1,\infty)$ it holds that

$$\begin{aligned} \mathfrak{e}_k &\le \left[\sup_{x\in D} \left|\left|\left|\big(W_{1,k} - W_{2,k}\big)\Big(\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x)\big)\Big)\right|\right|\right|\right] \\ &\quad + \left[\sup_{x\in D} \left|\left|\left| W_{3-j,k}\Big(\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{1,k-1}))(x)\big) - \mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{2,k-1}))(x)\big)\Big)\right|\right|\right|\right] + |||\theta_1 - \theta_2||| \\ &\le \left[\sup_{v\in\mathbb{R}^{l_{k-1}}\setminus\{0\}} \left(\frac{|||(W_{1,k} - W_{2,k})v|||}{|||v|||}\right)\right]\left[\sup_{x\in D} \left|\left|\left|\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x)\big)\right|\right|\right|\right] + |||\theta_1 - \theta_2||| \\ &\quad + \left[\sup_{v\in\mathbb{R}^{l_{k-1}}\setminus\{0\}} \left(\frac{|||W_{3-j,k}v|||}{|||v|||}\right)\right]\left[\sup_{x\in D} \left|\left|\left|\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{1,k-1}))(x)\big) - \mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{2,k-1}))(x)\big)\right|\right|\right|\right]. \end{aligned} \tag{97}$$

Lemma 2.35 and Lemma 2.34 therefore show that for all $j \in \{1,2\}$, $k \in \{1,2,\ldots,L\} \cap (1,\infty)$ it holds that

$$\begin{aligned} \mathfrak{e}_k &\le l_{k-1}\, |||\theta_1 - \theta_2|||\left[\sup_{x\in D} \left|\left|\left|\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x)\big)\right|\right|\right|\right] + |||\theta_1 - \theta_2||| \\ &\quad + l_{k-1}\, |||\theta_{3-j}|||\left[\sup_{x\in D} \left|\left|\left|\mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{1,k-1}))(x)\big) - \mathfrak{R}_{l_{k-1}}\big((\mathcal{R}_{\mathfrak{r}}(\phi_{2,k-1}))(x)\big)\right|\right|\right|\right] \\ &\le l_{k-1}\, |||\theta_1 - \theta_2|||\left[\sup_{x\in D} |||(\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x)|||\right] + |||\theta_1 - \theta_2||| \\ &\quad + l_{k-1}\, |||\theta_{3-j}|||\left[\sup_{x\in D} |||(\mathcal{R}_{\mathfrak{r}}(\phi_{1,k-1}))(x) - (\mathcal{R}_{\mathfrak{r}}(\phi_{2,k-1}))(x)|||\right] \\ &\le |||\theta_1 - \theta_2|||\,(l_{k-1}\, \mathfrak{m}_{j,k-1} + 1) + l_{k-1}\, |||\theta_{3-j}|||\, \mathfrak{e}_{k-1}. \end{aligned} \tag{98}$$

Hence, we obtain that for all $j \in \{1,2\}$, $k \in \{1,2,\ldots,L\} \cap (1,\infty)$ it holds that

$$\mathfrak{e}_k \le \mathfrak{m}_{j,k-1}\, |||\theta_1 - \theta_2|||\,(l_{k-1} + 1) + l_{k-1}\, |||\theta_{3-j}|||\, \mathfrak{e}_{k-1}. \tag{99}$$

Combining this with (95), the fact that $\mathfrak{e}_0 = 0$, and the fact that $\mathfrak{m}_{1,0} = \mathfrak{m}_{2,0}$ demonstrates that for all $j \in \{1, 2\}$, $k \in \{1, 2, \ldots, L\}$ it holds that

$$\mathfrak{e}_k \leq \mathfrak{m}_{j,k-1} \, (l_{k-1} + 1) \, |||\theta_1 - \theta_2||| + l_{k-1} \, |||\theta_{3-j}||| \, \mathfrak{e}_{k-1}. \tag{100}$$

This shows that for all $j = (j_n)_{n \in \{0,1,\ldots,L\}} \colon \{0, 1, \ldots, L\} \to \{1, 2\}$ and all $k \in \{1, 2, \ldots, L\}$ it holds that

$$\mathfrak{e}_k \leq \mathfrak{m}_{j_{k-1},k-1} \, (l_{k-1} + 1) \, |||\theta_1 - \theta_2||| + l_{k-1} \, |||\theta_{3-j_{k-1}}||| \, \mathfrak{e}_{k-1}. \tag{101}$$

Therefore, we obtain that for all $j = (j_n)_{n \in \{0,1,\ldots,L\}} \colon \{0, 1, \ldots, L\} \to \{1, 2\}$ and all $k \in \{1, 2, \ldots, L\}$ it holds that

$$\begin{aligned} \mathfrak{e}_k &\leq \sum_{n=0}^{k-1} \left(\left[\prod_{m=n+1}^{k-1} \left(l_m \, |||\theta_{3-j_m}||| \right) \right] \mathfrak{m}_{j_n,n} \, (l_n + 1) \, |||\theta_1 - \theta_2||| \right) \\ &= |||\theta_1 - \theta_2||| \left[\sum_{n=0}^{k-1} \left(\left[\prod_{m=n+1}^{k-1} \left(l_m \, |||\theta_{3-j_m}||| \right) \right] \mathfrak{m}_{j_n,n} \, (l_n + 1) \right) \right]. \end{aligned} \tag{102}$$

Next observe that Lemma 2.35 ensures that for all $j \in \{1, 2\}$, $k \in \{1, 2, \ldots, L\} \cap (1, \infty)$, $x \in D$ it holds that

$$\begin{aligned} |||(\mathcal{R}_{\mathfrak{r}}(\phi_{j,k}))(x)||| &= \left|\left|\left| W_{j,k} \Big(\mathfrak{R}_{l_{k-1}} \big((\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x) \big) \Big) + B_{j,k} \right|\right|\right| \\ &\leq \left[\sup_{v \in \mathbb{R}^{l_{k-1}} \setminus \{0\}} \frac{|||W_{j,k} v|||}{|||v|||} \right] \left|\left|\left| \mathfrak{R}_{l_{k-1}} \big((\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x) \big) \right|\right|\right| + |||B_{j,k}||| \\ &\leq l_{k-1} \, |||\theta_j||| \left|\left|\left| \mathfrak{R}_{l_{k-1}} \big((\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x) \big) \right|\right|\right| + |||\theta_j||| \\ &\leq l_{k-1} \, |||\theta_j||| \, |||(\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x)||| + |||\theta_j||| \\ &= \left(l_{k-1} \, |||(\mathcal{R}_{\mathfrak{r}}(\phi_{j,k-1}))(x)||| + 1 \right) |||\theta_j||| \\ &\leq \left(l_{k-1} \mathfrak{m}_{j,k-1} + 1 \right) |||\theta_j||| \leq \mathfrak{m}_{j,k-1} \, (l_{k-1} + 1) \, |||\theta_j|||. \end{aligned} \tag{103}$$

Hence, we obtain for all $j \in \{1, 2\}$, $k \in \{1, 2, \ldots, L\} \cap (1, \infty)$ that

$$\mathfrak{m}_{j,k} \leq \max\{1, \mathfrak{m}_{j,k-1} \, (l_{k-1} + 1) \, |||\theta_j|||\}. \tag{104}$$

Furthermore, note that Lemma 2.35 assures that for all $j \in \{1, 2\}$, $x \in D$ it holds that

$$\begin{aligned} |||(\mathcal{R}_{\mathfrak{r}}(\phi_{j,1}))(x)||| &= |||W_{j,1} x + B_{j,1}||| \\ &\leq \left[\sup_{v \in \mathbb{R}^{l_0} \setminus \{0\}} \frac{|||W_{j,1} v|||}{|||v|||} \right] |||x||| + |||B_{j,1}||| \\ &\leq l_0 \, |||\theta_j||| \, |||x||| + |||\theta_j||| \leq l_0 \, |||\theta_j||| \max\{|a|, |b|\} + |||\theta_j||| \\ &= (l_0 \max\{|a|, |b|\} + 1) \, |||\theta_j||| \leq \mathfrak{m}_{1,0} \, (l_0 + 1) \, |||\theta_j|||. \end{aligned} \tag{105}$$

Therefore, we obtain that for all $j \in \{1, 2\}$ it holds that

$$\mathfrak{m}_{j,1} \leq \max\{1, \mathfrak{m}_{j,0} \, (l_0 + 1) \, |||\theta_j|||\}. \tag{106}$$

Combining this with (104) demonstrates that for all $j \in \{1, 2\}$, $k \in \{1, 2, \ldots, L\}$ it holds that

$$\mathfrak{m}_{j,k} \leq \max\{1, \mathfrak{m}_{j,k-1} \, (l_{k-1} + 1) \, |||\theta_j|||\}. \tag{107}$$

Hence, we obtain that for all $j \in \{1, 2\}$, $k \in \{0, 1, \ldots, L\}$ it holds that

$$\mathfrak{m}_{j,k} \leq \mathfrak{m}_{j,0} \left[\prod_{n=0}^{k-1} (l_n + 1) \right] \left[\max\{1, |||\theta_j|||\} \right]^k. \tag{108}$$

Combining this with (102) proves that for all $j = (j_n)_{n\in\{0,1,\dots,L\}}\colon \{0,1,\dots,L\} \to \{1,2\}$ and all $k \in \{1,2,\dots,L\}$ it holds that

$$\begin{aligned}
\mathfrak{e}_k &\le |||\theta_1-\theta_2||| \left[\sum_{n=0}^{k-1}\left(\left[\prod_{m=n+1}^{k-1}\bigl(l_m\,|||\theta_{3-j_m}|||\bigr)\right]\left(\mathfrak{m}_{j_n,0}\left[\prod_{v=0}^{n-1}(l_v+1)\right]\max\{1,|||\theta_{j_n}|||^n\}\,(l_n+1)\right)\right)\right]\\
&= \mathfrak{m}_{1,0}\,|||\theta_1-\theta_2||| \left[\sum_{n=0}^{k-1}\left(\left[\prod_{m=n+1}^{k-1}\bigl(l_m\,|||\theta_{3-j_m}|||\bigr)\right]\left(\left[\prod_{v=0}^{n}(l_v+1)\right]\max\{1,|||\theta_{j_n}|||^n\}\right)\right)\right]\\
&\le \mathfrak{m}_{1,0}\,|||\theta_1-\theta_2||| \left[\sum_{n=0}^{k-1}\left(\left[\prod_{m=n+1}^{k-1}|||\theta_{3-j_m}|||\right]\left[\prod_{v=0}^{k-1}(l_v+1)\right]\max\{1,|||\theta_{j_n}|||^n\}\right)\right]\\
&= \mathfrak{m}_{1,0}\,|||\theta_1-\theta_2||| \left[\prod_{n=0}^{k-1}(l_n+1)\right]\left[\sum_{n=0}^{k-1}\left(\left[\prod_{m=n+1}^{k-1}|||\theta_{3-j_m}|||\right]\max\{1,|||\theta_{j_n}|||^n\}\right)\right].
\end{aligned}\tag{109}$$

Therefore, we obtain that for all $j \in \{1,2\}$, $k \in \{1,2,\dots,L\}$ it holds that

$$\begin{aligned}
\mathfrak{e}_k &\le \mathfrak{m}_{1,0}\,|||\theta_1-\theta_2||| \left[\prod_{n=0}^{k-1}(l_n+1)\right]\left[\sum_{n=0}^{k-1}\left(\left[\prod_{m=n+1}^{k-1}|||\theta_{3-j}|||\right]\max\{1,|||\theta_j|||^n\}\right)\right]\\
&= \mathfrak{m}_{1,0}\,|||\theta_1-\theta_2||| \left[\prod_{n=0}^{k-1}(l_n+1)\right]\left[\sum_{n=0}^{k-1}\left(\max\{1,|||\theta_j|||^n\}\,|||\theta_{3-j}|||^{k-1-n}\right)\right]\\
&\le k\,\mathfrak{m}_{1,0}\,|||\theta_1-\theta_2|||\,(\max\{1,|||\theta_1|||,|||\theta_2|||\})^{k-1}\left[\prod_{m=0}^{k-1}(l_m+1)\right].
\end{aligned}\tag{110}$$

The proof of Theorem 2.36 is thus completed. □

Corollary 2.37. *Let $a \in \mathbb{R}$, $b \in [a,\infty)$, $u \in [-\infty,\infty)$, $v \in (u,\infty]$, $d, L \in \mathbb{N}$, $l = (l_0, l_1, \dots, l_L) \in \mathbb{N}^{L+1}$ satisfy*

$$d \ge \sum_{k=1}^{L} l_k(l_{k-1}+1). \tag{111}$$

Then it holds for all $\theta, \vartheta \in \mathbb{R}^d$ that

$$\sup_{x\in[a,b]^{l_0}} |||\mathscr{N}^{\theta,l}_{u,v}(x) - \mathscr{N}^{\vartheta,l}_{u,v}(x)||| \le L\max\{1,|a|,|b|\}\,(|||l|||+1)^L\,(\max\{1,|||\theta|||,|||\vartheta|||\})^{L-1}\,|||\theta-\vartheta||| \tag{112}$$

(cf. Definitions 2.8 and 2.20).

Proof of Corollary 2.37. Observe that Theorem 2.36 and Lemma 2.33 demonstrate that for all $\theta, \vartheta \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
\sup_{x\in[a,b]^{l_0}} |||\mathscr{N}^{\theta,l}_{u,v}(x) - \mathscr{N}^{\vartheta,l}_{u,v}(x)||| &= \sup_{x\in[a,b]^{l_0}} \left|\left|\left|\mathfrak{C}_{u,v,l_L}\bigl((\mathscr{N}^{\theta,l}_{-\infty,\infty})(x)\bigr) - \mathfrak{C}_{u,v,l_L}\bigl((\mathscr{N}^{\vartheta,l}_{-\infty,\infty})(x)\bigr)\right|\right|\right|\\
&\le \sup_{x\in[a,b]^{l_0}} |||(\mathscr{N}^{\theta,l}_{-\infty,\infty})(x) - (\mathscr{N}^{\vartheta,l}_{-\infty,\infty})(x)|||\\
&\le L\max\{1,|a|,|b|\}\,(|||l|||+1)^L\,(\max\{1,|||\theta|||,|||\vartheta|||\})^{L-1}\,|||\theta-\vartheta|||
\end{aligned}\tag{113}$$

(cf. Definitions 2.8, 2.20, and 2.7). This completes the proof of Corollary 2.37. □

3 Separate analyses of the error sources

In this section we study separately the approximation error (see Section 3.1 below), the generalization error (see Section 3.2 below), and the optimization error (see Section 3.3 below).

In particular, the main result in Section 3.1, Proposition 3.5 below, establishes an upper bound for the error in the approximation of a Lipschitz continuous function by DNNs. This approximation result is obtained by combining the essentially well-known approximation result in Lemma 3.1 with the DNN calculus in Section 2.2 above (cf., e.g., Grohs et al. [31, 32]). Some of the results in Section 3.1 are partially based on material in publications from the scientific literature. In particular, the elementary result in Lemma 3.2 is basically well-known in the scientific literature. For further approximation results for DNNs we refer, e.g., to [1, 3, 4, 11, 12, 13, 14, 16, 17, 19, 21, 22, 23, 24, 25, 27, 29, 30, 31, 33, 34, 36, 38, 39, 40, 41, 42, 44, 47, 49, 52, 53, 54, 55, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 69, 70, 72, 73, 74] and the references mentioned therein.

In Lemmas 3.20 and 3.21 in Section 3.2 below we study the generalization error. Our analysis in Section 3.2 is in parts inspired by Berner et al. [10] and Cucker & Smale [18]. Proposition 3.10 in Section 3.2.1 is known as Hoeffding's inequality in the scientific literature and Proposition 3.10 is, e.g., proved as Theorem 2 in Hoeffding [37]. The proof of Proposition 3.12 can be found, e.g., in Cucker & Smale [18, Proposition 5] (cf. also Berner et al. [10, Proposition 4.3]). For further results on the generalization error we refer, e.g., to [5, 35, 51, 68, 71] and the references mentioned therein.

In the two elementary results in Section 3.3, Lemmas 3.22 and 3.23, we study the optimization error of the minimum Monte Carlo algorithm. A related result can be found, e.g., in [6, Lemma 3.5]. For further results on the optimization error we refer, e.g., to [2, 9, 15, 20, 26, 43, 45, 46, 48] and the references mentioned therein.

3.1 Analysis of the approximation error

3.1.1 Approximations for Lipschitz continuous functions

Lemma 3.1. *Let (E, δ) be a metric space, let $\mathcal{M} \subseteq E$ satisfy $\mathcal{M} \neq \emptyset$, let $L \in [0, \infty)$, let $f \colon E \to \mathbb{R}$ satisfy for all $x \in E$, $y \in \mathcal{M}$ that $|f(x) - f(y)| \leq L\delta(x, y)$, and let $F \colon E \to \mathbb{R} \cup \{\infty\}$ satisfy for all $x \in E$ that*

$$F(x) = \sup_{y \in \mathcal{M}} [f(y) - L\delta(x, y)]. \tag{114}$$

Then

(i) it holds for all $x \in E$ that $F(x) \leq f(x)$,

(ii) it holds for all $x \in \mathcal{M}$ that $F(x) = f(x)$,

(iii) it holds for all $x, y \in E$ that $|F(x) - F(y)| \leq L\delta(x, y)$, and

(iv) it holds for all $x \in E$ that

$$|F(x) - f(x)| \leq 2L \left[\inf_{y \in \mathcal{M}} \delta(x, y) \right]. \tag{115}$$

Proof of Lemma 3.1. First, observe that the hypothesis that for all $x \in E$, $y \in \mathcal{M}$ it holds that $|f(x) - f(y)| \leq L\delta(x, y)$ ensures that for all $x \in E$, $y \in \mathcal{M}$ it holds that

$$f(x) \geq f(y) - L\delta(x, y). \tag{116}$$

Hence, we obtain that for all $x \in E$ it holds that

$$f(x) \geq \sup_{y \in \mathcal{M}} [f(y) - L\delta(x, y)] = F(x). \tag{117}$$

This establishes item (i). Next observe that (114) implies that for all $x \in \mathcal{M}$ it holds that

$$F(x) \geq f(x) - L\delta(x, x) = f(x). \tag{118}$$

Combining this with item (i) establishes item (ii). In the next step we note that (114) and the fact that for all $x \in E$ it holds that $F(x) \leq f(x) < \infty$ show that for all $x, y \in E$ it holds that

$$\begin{aligned} F(x) - F(y) &= \left[\sup_{v \in \mathcal{M}} (f(v) - L\delta(x, v))\right] - \left[\sup_{w \in \mathcal{M}} (f(w) - L\delta(y, w))\right] \\ &= \sup_{v \in \mathcal{M}} \left[f(v) - L\delta(x, v) - \sup_{w \in \mathcal{M}} (f(w) - L\delta(y, w))\right] \\ &\leq \sup_{v \in \mathcal{M}} \left[f(v) - L\delta(x, v) - (f(v) - L\delta(y, v))\right] \\ &= L\left[\sup_{v \in \mathcal{M}} (\delta(y, v) - \delta(x, v))\right] \\ &\leq L\left[\sup_{v \in \mathcal{M}} (\delta(y, x) + \delta(x, v) - \delta(x, v))\right] = L\delta(x, y). \end{aligned} \tag{119}$$

Combining this with the fact that for all $x, y \in E$ it holds that $\delta(x, y) = \delta(y, x)$ establishes item (iii). Observe that item (ii), the triangle inequality, item (iii), and the hypothesis that for all $x \in E$, $y \in \mathcal{M}$ it holds that $|f(x) - f(y)| \leq L\delta(x, y)$ ensure that for all $x \in E$ it holds that

$$\begin{aligned} |F(x) - f(x)| &= \inf_{y \in \mathcal{M}} |F(x) - F(y) + f(y) - f(x)| \\ &\leq \inf_{y \in \mathcal{M}} (|F(x) - F(y)| + |f(y) - f(x)|) \\ &\leq \inf_{y \in \mathcal{M}} (2L\delta(x, y)) = 2L\left[\inf_{y \in \mathcal{M}} \delta(x, y)\right]. \end{aligned} \tag{120}$$

This establishes item (iv). The proof of Lemma 3.1 is thus completed. □

3.1.2 DNN representations for maxima

Lemma 3.2. *Let* $\Phi \in \mathbf{N}$ *satisfy*

$$\Phi = \left(\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}, 0 \right) \right) \in \left((\mathbb{R}^{3 \times 2} \times \mathbb{R}^3) \times (\mathbb{R}^{1 \times 3} \times \mathbb{R}) \right) \tag{121}$$

(cf. Definition 2.9). Then

(i) it holds for all $k \in \mathbb{N}$ *that* $\mathcal{L}(\mathfrak{I}_k) = 2$,

(ii) there exist unique $\phi_k \in \mathbf{N}$, $k \in \{2, 3, \ldots\}$, *which satisfy for all* $k \in \{2, 3, \ldots\}$ *that* $\phi_2 = \Phi$, $\mathcal{I}(\phi_k) = \mathcal{O}(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1}))$, *and*

$$\phi_{k+1} = \phi_k \bullet \left(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1})\right), \tag{122}$$

(iii) it holds for all $k \in \{2, 3, \ldots\}$ *that* $\mathcal{L}(\phi_k) = k$, *and*

(iv) it holds for all $k \in \{2, 3, \ldots\}$ *that* $\mathcal{D}(\phi_k) = (k, 2k - 1, 2k - 3, \ldots, 3, 1) \in \mathbb{N}^{k+1}$, *and*

(v) it holds for all $k \in \{2, 3, \ldots\}$, $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ *that*

$$\left(\mathcal{R}_{\mathfrak{r}}(\phi_k)\right)(x) = \max\{x_1, x_2, \ldots, x_k\} \tag{123}$$

(cf. Definitions 2.18, 2.16, 2.19, 2.4, and 2.10).

Proof of Lemma 3.2. First, note that, e.g., item (i) in [32, Lemma 3.16] shows that for all $k \in \mathbb{N}$ it holds that

$$\mathcal{D}(\mathfrak{I}_k) = (k, 2k, k). \tag{124}$$

This establishes item (i). Next note that (121) demonstrates that

$$\mathcal{D}(\Phi) = (2, 3, 1). \tag{125}$$

Combining this and (124) with item (i) in [31, Proposition 2.20] shows that for all $k \in \mathbb{N}$ it holds that

$$\mathcal{D}(\mathbf{P}_2(\Phi, \mathfrak{I}_k)) = (k + 2, 2k + 3, k + 1) \tag{126}$$

(cf. Definition 2.16). Hence, we obtain that for all $k \in \{2, 3, \ldots\}$ it holds that

$$\mathcal{D}(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1})) = (k + 1, 2k + 1, k). \tag{127}$$

Combining this with (126) ensures that for all $k \in \{2, 3, \ldots\}$ it holds that

$$\mathcal{O}(\mathbf{P}_2(\Phi, \mathfrak{I}_k)) = k + 1 = \mathcal{I}(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1})). \tag{128}$$

Moreover, note that (121) and (126) assure that

$$\mathcal{I}(\Phi) = 2 = \mathcal{O}(\mathbf{P}_2(\Phi, \mathfrak{I}_1)). \tag{129}$$

Furthermore, observe that item (i) in [31, Proposition 2.6] and (128) show that for all $k \in \{2, 3, \ldots\}$, $\psi \in \mathbf{N}$ with $\mathcal{I}(\psi) = \mathcal{O}(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1}))$ it holds that

$$\mathcal{I}\big(\psi \bullet \big(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1})\big)\big) = \mathcal{I}(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1})) = \mathcal{O}(\mathbf{P}_2(\Phi, \mathfrak{I}_k)) \tag{130}$$

(cf. Definition 2.19). Combining this and (129) with induction establishes item (ii). In the next step we note that (122) and item (ii) in [31, Proposition 2.6] imply that for all $k \in \{2, 3, \ldots\}$ it holds that

$$\mathcal{L}(\phi_{k+1}) = \mathcal{L}(\phi_k) + \mathcal{L}(\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1})) - 1 = \mathcal{L}(\phi_k) + 1. \tag{131}$$

Combining this and the fact that $\mathcal{L}(\phi_2) = 2$ with induction establishes item (iii). Furthermore, observe that (122), (127), and item (i) in [31, Proposition 2.6] demonstrate that for all $k \in \{2, 3, \ldots\}$, $l_0, l_1, \ldots, l_k \in \mathbb{N}$ with $\mathcal{D}(\phi_k) = (l_0, l_1, \ldots, l_k)$ it holds that

$$\mathcal{D}(\phi_{k+1}) = \mathcal{D}\big(\phi_k \bullet \big(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1})\big)\big) = (k + 1, 2k + 1, l_1, l_2, \ldots, l_k). \tag{132}$$

This, item (iii), the fact that $\mathcal{D}(\phi_2) = (2, 3, 1)$, and induction establish item (iv). Moreover, note that (121) ensures that for all $(x_1, x_2) \in \mathbb{R}^2$ it holds that

$$\begin{aligned}
\big(\mathcal{R}_{\mathfrak{r}}(\Phi)\big)(x_1, x_2) &= \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \left(\mathfrak{M}_{\mathfrak{r},3} \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) \right) + 0 \\
&= \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \max\{x_1 - x_2, 0\} \\ \max\{x_2, 0\} \\ \max\{-x_2, 0\} \end{pmatrix} \\
&= \max\{x_1 - x_2, 0\} + \max\{x_2, 0\} - \max\{-x_2, 0\} \\
&= \max\{x_1 - x_2, 0\} + x_2 = \max\{x_1, x_2\}
\end{aligned} \tag{133}$$

(cf. Definitions 2.4, 2.10, and 2.3). Combining this and item (iii) in [32, Lemma 3.16] with [31, Proposition 2.19] proves that for all $k \in \{2, 3, \ldots\}$, $x = (x_1, x_2, \ldots, x_{k+1}) \in \mathbb{R}^{k+1}$ it holds that

$$\begin{aligned} \big(\mathcal{R}_{\mathfrak{r}}(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1}))\big)(x) &= \big(\big(\mathcal{R}_{\mathfrak{r}}(\Phi)\big)(x_1, x_2), \big(\mathcal{R}_{\mathfrak{r}}(\mathfrak{I}_{k-1})\big)(x_3, x_4, \ldots, x_{k+1})\big) \\ &= (\max\{x_1, x_2\}, x_3, x_4, \ldots, x_{k+1}) \in \mathbb{R}^k. \end{aligned} \tag{134}$$

Item (v) in [31, Proposition 2.6] and (122) hence show that for all $k \in \{2, 3, \ldots\}$, $x = (x_1, x_2, \ldots, x_{k+1}) \in \mathbb{R}^{k+1}$ it holds that

$$\begin{aligned} \big(\mathcal{R}_{\mathfrak{r}}(\phi_{k+1})\big)(x) &= \Big(\mathcal{R}_{\mathfrak{r}}\big(\phi_k \bullet \big(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1})\big)\big)\Big)(x) = \big(\big[\mathcal{R}_{\mathfrak{r}}(\phi_k)\big] \circ \big[\mathcal{R}_{\mathfrak{r}}(\mathbf{P}_2(\Phi, \mathfrak{I}_{k-1}))\big]\big)(x) \\ &= \big(\mathcal{R}_{\mathfrak{r}}(\phi_k)\big)(\max\{x_1, x_2\}, x_3, x_4, \ldots, x_{k+1}). \end{aligned} \tag{135}$$

This, the fact that $\phi_2 = \Phi$, (133), and induction establish item (v). The proof of Lemma 3.2 is thus completed. □

Lemma 3.3. *Let $A_k \in \mathbb{R}^{(2k-1)\times k}$, $k \in \{2, 3, \ldots\}$, and $C_k \in \mathbb{R}^{(k-1)\times(2k-1)}$, $k \in \{2, 3, \ldots\}$, satisfy for all $k \in \{2, 3, \ldots\}$ that*

$$A_k = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \qquad \text{and} \qquad C_k = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \tag{136}$$

and let $\phi_k = ((W_{k,1}, B_{k,1}), (W_{k,2}, B_{k,2}), \ldots, (W_{k,k}, B_{k,k})) \in \mathbf{N}$, $k \in \{2, 3, \ldots\}$, satisfy for all $k \in \{2, 3, \ldots\}$ that $\mathcal{I}(\phi_k) = \mathcal{O}(\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1}))$, $\phi_{k+1} = \phi_k \bullet (\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1}))$, and

$$\phi_2 = \left(\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}, 0 \right) \right) \in \big((\mathbb{R}^{3\times 2} \times \mathbb{R}^3) \times (\mathbb{R}^{1\times 3} \times \mathbb{R})\big) \tag{137}$$

(cf. Definitions 2.9, 2.18, 2.19, and 2.16 and Lemma 3.2). Then

(i) it holds for all $k \in \{2, 3, \ldots\}$ that $W_{k,1} = A_k$,

(ii) it holds for all $k \in \{2, 3, \ldots\}$, $l \in \{1, 2, \ldots, k\}$ that $B_{k,l} = 0 \in \mathbb{R}^{2(k-l)+1}$,

(iii) it holds for all $k \in \{2, 3, \ldots\}$, $l \in \{3, 4, \ldots, k+1\}$ that $(W_{k+1,l}, B_{k+1,l}) = (W_{k,l-1}, B_{k,l-1})$,

(iv) it holds for all $k \in \{2, 3, \ldots\}$ that $W_{k+1,2} = W_{k,1}C_{k+1}$,

(v) it holds for all $k \in \{2, 3, \ldots\}$ that $(0, 0, \ldots, 0) \neq \mathcal{T}(\phi_k) \in \big(\{-1, 0, 1\}^{\mathcal{P}(\phi_k)}\big)$, and

(vi) it holds for all $k \in \{2, 3, \ldots\}$ that $|||\mathcal{T}(\phi_k)||| = 1$

(cf. Definitions 2.11 and 2.20).

Proof of Lemma 3.3. First, note that (28), (29), (136), and (137) ensure that for all $k \in \{2, 3, \ldots\}$ it holds that

$$\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1}) = (\mathfrak{N}_{A_{k+1}}, \mathfrak{N}_{C_{k+1}}) \tag{138}$$

(cf. Definition 2.17). This and (137) imply that for all $k \in \{2,3,\ldots\}$ it holds that

$$\begin{aligned}\phi_{k+1} &= \phi_k \bullet \big(\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1})\big) \\ &= ((W_{k,1}, B_{k,1}), (W_{k,2}, B_{k,2}), \ldots, (W_{k,k}, B_{k,k})) \bullet (\mathfrak{N}_{A_{k+1}}, \mathfrak{N}_{C_{k+1}}) \\ &= (\mathfrak{N}_{A_{k+1}}, (W_{k,1}C_{k+1}, B_{k,1}), (W_{k,2}, B_{k,2}), \ldots, (W_{k,k}, B_{k,k})).\end{aligned} \tag{139}$$

This, (136), and (137) establish item (i). Next observe that (137), (139), item (iv) in Lemma 3.2, and induction prove item (ii). Moreover, note that (139) establishes items (iii) and (iv). In addition, observe that item (i) proves that for all $k \in \{2,3,\ldots\}$ it holds that

$$\begin{aligned}W_{k,1}C_{k+1} = A_kC_{k+1} &= \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.\end{aligned} \tag{140}$$

Combining this, (137), and (139) with induction proves item (v). Next note that item (v) establishes item (vi). The proof of Lemma 3.3 is thus completed. ∎

3.1.3 Interpolations through DNNs

Lemma 3.4. *Let $\phi_k \in \mathbf{N}$, $k \in \{2,3,\ldots\}$, satisfy for all $k \in \{2,3,\ldots\}$ that $\mathcal{I}(\phi_k) = \mathcal{O}(\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1}))$, $\phi_{k+1} = \phi_k \bullet (\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1}))$, and*

$$\phi_2 = \left(\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}, 0 \right) \right) \in \big((\mathbb{R}^{3\times 2} \times \mathbb{R}^3) \times (\mathbb{R}^{1\times 3} \times \mathbb{R}) \big), \tag{141}$$

let $d \in \mathbb{N}$, $L \in [0,\infty)$, let $\mathcal{M} \subseteq \mathbb{R}^d$ satisfy $|\mathcal{M}| \in \{2,3,\ldots\}$, let $m\colon \{1,2,\ldots,|\mathcal{M}|\} \to \mathcal{M}$ be bijective, let $f\colon \mathcal{M} \to \mathbb{R}$ and $F\colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that

$$F(x) = \max_{y=(y_1,y_2\ldots,y_d)\in\mathcal{M}} \left[f(y) - L\left(\sum_{i=1}^d |x_i - y_i| \right) \right], \tag{142}$$

let $W_1 \in \mathbb{R}^{(2d)\times d}$, $W_2 \in \mathbb{R}^{1\times(2d)}$, and $B_z \in \mathbb{R}^{2d}$, $z \in \mathcal{M}$, satisfy for all $z = (z_1, z_2, \ldots, z_d) \in \mathcal{M}$ that

$$W_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix}, \qquad B_z = \begin{pmatrix} -z_1 \\ z_1 \\ -z_2 \\ z_2 \\ \vdots \\ -z_d \\ z_d \end{pmatrix}, \qquad \text{and} \qquad W_2 = \begin{pmatrix} -L & -L & \cdots & -L \end{pmatrix}, \tag{143}$$

let $\mathcal{W}_1 \in \mathbb{R}^{(2d|\mathcal{M}|)\times d}$, $\mathcal{B}_1 \in \mathbb{R}^{2d|\mathcal{M}|}$, $\mathcal{W}_2 \in \mathbb{R}^{|\mathcal{M}|\times(2d|\mathcal{M}|)}$, $\mathcal{B}_2 \in \mathbb{R}^{|\mathcal{M}|}$ *satisfy*

$$\mathcal{W}_1 = \begin{pmatrix} W_1 \\ W_1 \\ \vdots \\ W_1 \end{pmatrix}, \quad \mathcal{B}_1 = \begin{pmatrix} B_{m(1)} \\ B_{m(2)} \\ \vdots \\ B_{m(|\mathcal{M}|)} \end{pmatrix}, \quad \mathcal{W}_2 = \begin{pmatrix} W_2 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_2 \end{pmatrix}, \quad \text{and} \quad \mathcal{B}_2 = \begin{pmatrix} f(m(1)) \\ f(m(2)) \\ \vdots \\ f(m(|\mathcal{M}|)) \end{pmatrix}, \tag{144}$$

and let $\Phi \in \mathbf{N}$ *satisfy* $\Phi = \phi_{|\mathcal{M}|} \bullet ((\mathcal{W}_1, \mathcal{B}_1), (\mathcal{W}_2, \mathcal{B}_2))$ *(cf. Definitions 2.9, 2.18, 2.16, and 2.19 and Lemma 3.2). Then*

(i) it holds that $\mathcal{D}(\Phi) = (d, 2d|\mathcal{M}|, 2|\mathcal{M}| - 1, 2|\mathcal{M}| - 3, \ldots, 3, 1) \in \mathbb{N}^{|\mathcal{M}|+2}$,

(ii) it holds that $\mathcal{L}(\Phi) = |\mathcal{M}| + 1$,

(iii) it holds that $|||\mathcal{T}(\Phi)||| \leq \max\{1, L, \sup_{z\in\mathcal{M}} |||z|||, 2[\sup_{z\in\mathcal{M}} |f(z)|]\}$, *and*

(iv) it holds that $F = \mathcal{R}_{\mathfrak{r}}(\Phi)$

(cf. Definitions 2.11, 2.20, 2.4, and 2.10).

Proof of Lemma 3.4. Throughout this proof let $\Psi \in \mathbf{N}$ satisfy $\Psi = ((\mathcal{W}_1, \mathcal{B}_1), (\mathcal{W}_2, \mathcal{B}_2))$ and let $\mathfrak{m}_{i,j} \in \mathbb{R}$, $i \in \{1, 2, \ldots, |\mathcal{M}|\}$, $j \in \{1, 2 \ldots, d\}$, satisfy for all $i \in \{1, 2, \ldots, |\mathcal{M}|\}$, $j \in \{1, 2 \ldots, d\}$ that $m(i) = (\mathfrak{m}_{i,1}, \mathfrak{m}_{i,2}, \ldots, \mathfrak{m}_{i,d})$. Note that Lemma 3.2 establishes that there exist $\mathfrak{W}_1 \in \mathbb{R}^{(2|\mathcal{M}|-1)\times|\mathcal{M}|}$, $\mathfrak{B}_1 \in \mathbb{R}^{2|\mathcal{M}|-1}$, $\mathfrak{W}_2 \in \mathbb{R}^{(2|\mathcal{M}|-3)\times(2|\mathcal{M}|-1)}$, $\mathfrak{B}_2 \in \mathbb{R}^{2|\mathcal{M}|-3}$, $\ldots$, $\mathfrak{W}_{|\mathcal{M}|-1} \in \mathbb{R}^{3\times 5}$, $\mathfrak{B}_{|\mathcal{M}|-1} \in \mathbb{R}^3$, $\mathfrak{W}_{|\mathcal{M}|} \in \mathbb{R}^{1\times 3}$, $\mathfrak{B}_{|\mathcal{M}|} \in \mathbb{R}$ such that

$$\phi_{|\mathcal{M}|} = ((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_{|\mathcal{M}|}, \mathfrak{B}_{|\mathcal{M}|})). \tag{145}$$

Next observe that (144) establishes that $\mathcal{L}(\Psi) = 2$ and

$$\mathcal{D}(\Psi) = (d, 2d|\mathcal{M}|, |\mathcal{M}|). \tag{146}$$

Moreover, note that item (iv) in Lemma 3.2 ensures that

$$\mathcal{D}(\phi_{|\mathcal{M}|}) = (|\mathcal{M}|, 2|\mathcal{M}| - 1, 2|\mathcal{M}| - 3, \ldots, 3, 1) \in \mathbb{N}^{|\mathcal{M}|+1}. \tag{147}$$

This, the fact that $\Phi = \phi_{|\mathcal{M}|} \bullet \Psi$, (146), and item (i) in [31, Proposition 2.6] show that $\mathcal{L}(\Phi) = |\mathcal{M}| + 1$ and

$$\mathcal{D}(\Phi) = (d, 2d|\mathcal{M}|, 2|\mathcal{M}| - 1, 2|\mathcal{M}| - 3, \ldots, 3, 1) \in \mathbb{N}^{|\mathcal{M}|+2}. \tag{148}$$

This establishes items (i) and (ii). In the next step we note that the hypothesis that $\Phi = \phi_{|\mathcal{M}|} \bullet ((\mathcal{W}_1, \mathcal{B}_1), (\mathcal{W}_2, \mathcal{B}_2))$ and (145) ensure that

$$\begin{aligned} \Phi &= ((\mathfrak{W}_1, \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_{|\mathcal{M}|}, \mathfrak{B}_{|\mathcal{M}|})) \bullet ((\mathcal{W}_1, \mathcal{B}_1), (\mathcal{W}_2, \mathcal{B}_2)) \\ &= ((\mathcal{W}_1, \mathcal{B}_1), (\mathfrak{W}_1\mathcal{W}_2, \mathfrak{W}_1\mathcal{B}_2 + \mathfrak{B}_1), (\mathfrak{W}_2, \mathfrak{B}_2), \ldots, (\mathfrak{W}_{|\mathcal{M}|}, \mathfrak{B}_{|\mathcal{M}|})). \end{aligned} \tag{149}$$

Lemma 2.13 hence implies that

$$\mathcal{T}(\Phi) = \big(\mathcal{T}\big(((\mathcal{W}_1, \mathcal{B}_1))\big), \mathcal{T}\big(((\mathfrak{W}_1\mathcal{W}_2, \mathfrak{W}_1\mathcal{B}_2 + \mathfrak{B}_1))\big), \mathcal{T}\big(((\mathfrak{W}_2, \mathfrak{B}_2))\big), \ldots, \mathcal{T}\big(((\mathfrak{W}_{|\mathcal{M}|}, \mathfrak{B}_{|\mathcal{M}|}))\big)\big) \tag{150}$$

(cf. Definition 2.11). Moreover, note that (144) and item (i) in Lemma 3.3 imply that

$$\mathfrak{W}_1\mathcal{W}_2 = \underbrace{\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}}_{\in\mathbb{R}^{(2|\mathcal{M}|-1)\times|\mathcal{M}|}} \mathcal{W}_2 = \underbrace{\begin{pmatrix} W_2 & -W_2 & 0 & \cdots & 0 \\ 0 & W_2 & 0 & \cdots & 0 \\ 0 & -W_2 & 0 & \cdots & 0 \\ 0 & 0 & W_2 & \cdots & 0 \\ 0 & 0 & -W_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & W_2 \\ 0 & 0 & 0 & \cdots & -W_2 \end{pmatrix}}_{\in\mathbb{R}^{(2|\mathcal{M}|-1)\times(2d|\mathcal{M}|)}}. \tag{151}$$

In addition, observe that (144) and items (i) and (ii) in Lemma 3.3 show that

$$
\begin{aligned}
\mathfrak{W}_1\mathcal{B}_2+\mathfrak{B}_1 &= \underbrace{\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}}_{\in \mathbb{R}^{(2|\mathcal{M}|-1)\times|\mathcal{M}|}} \mathcal{B}_2 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}}_{\in \mathbb{R}^{2|\mathcal{M}|-1}} \\
&= \underbrace{\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}}_{\in \mathbb{R}^{(2|\mathcal{M}|-1)\times|\mathcal{M}|}} \underbrace{\begin{pmatrix} f(m(1)) \\ f(m(2)) \\ \vdots \\ f(m(|\mathcal{M}|)) \end{pmatrix}}_{\in \mathbb{R}^{|\mathcal{M}|}} = \underbrace{\begin{pmatrix} f(m(1)) - f(m(2)) \\ f(m(2)) \\ -f(m(2)) \\ f(m(3)) \\ -f(m(3)) \\ \vdots \\ f(m(|\mathcal{M}|)) \\ -f(m(|\mathcal{M}|)) \end{pmatrix}}_{\in \mathbb{R}^{2|\mathcal{M}|-1}}.
\end{aligned} \tag{152}
$$

This and (151) demonstrate that

$$
\begin{aligned}
&\left|\!\left|\!\left| \mathcal{T}\big(((\mathfrak{W}_1\mathcal{W}_2, \mathfrak{W}_1\mathcal{B}_2+\mathfrak{B}_1))\big) \right|\!\right|\!\right| \\
&= \max\{L, |f(m(1)) - f(m(2))|, |f(m(2))|, |f(m(3))|, \ldots, |f(m(|\mathcal{M}|))|\} \leq \max\left\{L, 2\left[\sup_{z\in\mathcal{M}} |f(z)|\right]\right\}
\end{aligned} \tag{153}
$$

(cf. Definition 2.20). Combining this, (144), and item (vi) in Lemma 3.3 with (150) proves that

$$
\begin{aligned}
|||\mathcal{T}(\Phi)||| &\leq \max\big\{\left|\!\left|\!\left|\mathcal{T}\big(((\mathcal{W}_1, \mathcal{B}_1))\big)\right|\!\right|\!\right|, \left|\!\left|\!\left|\mathcal{T}\big(((\mathfrak{W}_1\mathcal{W}_2, \mathfrak{W}_1\mathcal{B}_2+\mathfrak{B}_1))\big)\right|\!\right|\!\right|, |||\mathcal{T}(\phi_{|\mathcal{M}|})|||\big\} \\
&\leq \max\left\{1, \sup_{z\in\mathcal{M}} |||z|||, L, 2\left[\sup_{z\in\mathcal{M}} |f(z)|\right]\right\}.
\end{aligned} \tag{154}
$$

This establishes item (iii). Observe that (143) ensures that for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, $z = (z_1, z_2, \ldots, z_d) \in \mathcal{M}$ it holds that

$$
W_1 x + B_z = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix}}_{\in \mathbb{R}^{(2d)\times d}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} + \begin{pmatrix} -z_1 \\ z_1 \\ -z_2 \\ z_2 \\ \vdots \\ -z_d \\ z_d \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \\ x_2 \\ -x_2 \\ \vdots \\ x_d \\ -x_d \end{pmatrix} + \begin{pmatrix} -z_1 \\ z_1 \\ -z_2 \\ z_2 \\ \vdots \\ -z_d \\ z_d \end{pmatrix} = \begin{pmatrix} x_1 - z_1 \\ -(x_1 - z_1) \\ x_2 - z_2 \\ -(x_2 - z_2) \\ \vdots \\ x_d - z_d \\ -(x_d - z_d) \end{pmatrix}. \tag{155}
$$

This and (144) prove that for all $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $z = (z_1, z_2, \dots, z_d) \in \mathcal{M}$ it holds that

$$\begin{aligned} W_2\big(\mathfrak{R}_{2d}(W_1 x + B_z)\big) &= \underbrace{\begin{pmatrix} -L & -L & \cdots & -L \end{pmatrix}}_{\in \mathbb{R}^{1\times(2d)}} \begin{pmatrix} \max\{x_1 - z_1, 0\} \\ \max\{z_1 - x_1, 0\} \\ \max\{x_2 - z_2, 0\} \\ \max\{z_2 - x_2, 0\} \\ \vdots \\ \max\{x_d - z_d, 0\} \\ \max\{z_d - x_d, 0\} \end{pmatrix} \\ &= -L\left[\sum_{i=1}^d (\max\{x_i - z_i, 0\} + \max\{z_i - x_i, 0\})\right] = -L\left[\sum_{i=1}^d |x_i - z_i|\right] \end{aligned} \tag{156}$$

(cf. Definition 2.5). Moreover, note that (144) implies that for all $x \in \mathbb{R}^d$ it holds that

$$\mathcal{W}_1 x + \mathcal{B}_1 = \begin{pmatrix} W_1 x + B_{m(1)} \\ W_1 x + B_{m(2)} \\ \vdots \\ W_1 x + B_{m(|\mathcal{M}|)} \end{pmatrix}. \tag{157}$$

Therefore, we obtain that for all $x \in \mathbb{R}^d$ it holds that

$$\mathfrak{R}_{2d|\mathcal{M}|}(\mathcal{W}_1 x + \mathcal{B}_1) = \begin{pmatrix} \mathfrak{R}_{2d}(W_1 x + B_{m(1)}) \\ \mathfrak{R}_{2d}(W_1 x + B_{m(2)}) \\ \vdots \\ \mathfrak{R}_{2d}(W_1 x + B_{m(|\mathcal{M}|)}) \end{pmatrix}. \tag{158}$$

This, (144), and (156) imply that for all $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \big(\mathcal{R}_\mathfrak{r}(\Psi)\big)(x) &= \mathcal{W}_2\big(\mathfrak{R}_{2d|\mathcal{M}|}(\mathcal{W}_1 x + \mathcal{B}_1)\big) + \mathcal{B}_2 \\ &= \begin{pmatrix} W_2 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_2 \end{pmatrix} \begin{pmatrix} \mathfrak{R}_{2d}(W_1 x + B_{m(1)}) \\ \mathfrak{R}_{2d}(W_1 x + B_{m(2)}) \\ \vdots \\ \mathfrak{R}_{2d}(W_1 x + B_{m(|\mathcal{M}|)}) \end{pmatrix} + \begin{pmatrix} f(m(1)) \\ f(m(2)) \\ \vdots \\ f(m(|\mathcal{M}|)) \end{pmatrix} \\ &= \begin{pmatrix} W_2\big(\mathfrak{R}_{2d}(W_1 x + B_{m(1)})\big) \\ W_2\big(\mathfrak{R}_{2d}(W_1 x + B_{m(2)})\big) \\ \vdots \\ W_2\big(\mathfrak{R}_{2d}(W_1 x + B_{m(|\mathcal{M}|)})\big) \end{pmatrix} + \begin{pmatrix} f(m(1)) \\ f(m(2)) \\ \vdots \\ f(m(|\mathcal{M}|)) \end{pmatrix} \\ &= \begin{pmatrix} f(m(1)) - L\big[\sum_{i=1}^d |x_i - \mathfrak{m}_{1,i}|\big] \\ f(m(2)) - L\big[\sum_{i=1}^d |x_i - \mathfrak{m}_{2,i}|\big] \\ \vdots \\ f(m(|\mathcal{M}|)) - L\big[\sum_{i=1}^d |x_i - \mathfrak{m}_{|\mathcal{M}|,i}|\big] \end{pmatrix} \end{aligned} \tag{159}$$

(cf. Definitions 2.4 and 2.10). This, the fact that $\Phi = \phi_{|\mathcal{M}|} \bullet \Psi$, item (v) in Lemma 3.2, and item (v) in [31, Proposition 2.6] ensure that for all $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \big(\mathcal{R}_\mathfrak{r}(\Phi)\big)(x) &= \big(\big[\mathcal{R}_\mathfrak{r}(\phi_{|\mathcal{M}|})\big] \circ \big[\mathcal{R}_\mathfrak{r}(\Psi)\big]\big)(x) = \max_{i \in \{1,2,\dots,|\mathcal{M}|\}} \left[f(m(i)) - L\left(\sum_{j=1}^d |x_j - \mathfrak{m}_{i,j}|\right)\right] \\ &= \max_{y=(y_1,y_2\dots,y_d)\in\mathcal{M}} \left[f(z) - L\left(\sum_{i=1}^d |x_i - y_i|\right)\right]. \end{aligned} \tag{160}$$

This establishes item (iv). The proof of Lemma 3.4 is thus completed. ☐

3.1.4 Explicit approximations through DNNs

Proposition 3.5. *Let $\phi_k \in \mathbf{N}$, $k \in \{2, 3, \ldots\}$, satisfy for all $k \in \{2, 3, \ldots\}$ that $\mathcal{I}(\phi_k) = \mathcal{O}(\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1}))$, $\phi_{k+1} = \phi_k \bullet (\mathbf{P}_2(\phi_2, \mathfrak{I}_{k-1}))$, and*

$$\phi_2 = \left(\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}, 0 \right) \right) \in \big((\mathbb{R}^{3\times 2} \times \mathbb{R}^3) \times (\mathbb{R}^{1\times 3} \times \mathbb{R}) \big), \tag{161}$$

let $d \in \mathbb{N}$, $L \in \mathbb{R}$, let $D \subseteq \mathbb{R}^d$ be a set, let $f \colon D \to \mathbb{R}$ satisfy for all $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d) \in D$ that $|f(x) - f(y)| \leq L\big[\sum_{i=1}^d |x_i - y_i|\big]$, let $\mathcal{M} \subseteq D$ satisfy $|\mathcal{M}| \in \{2, 3, \ldots\}$, let $m \colon \{1, 2, \ldots, |\mathcal{M}|\} \to \mathcal{M}$ be bijective, let $W_1 \in \mathbb{R}^{(2d)\times d}$, $W_2 \in \mathbb{R}^{1\times(2d)}$, and $B_z \in \mathbb{R}^{2d}$, $z \in \mathcal{M}$, satisfy for all $z = (z_1, z_2, \ldots, z_d) \in \mathcal{M}$ that

$$W_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 \end{pmatrix}, \qquad B_z = \begin{pmatrix} -z_1 \\ z_1 \\ -z_2 \\ z_2 \\ \vdots \\ -z_d \\ z_d \end{pmatrix}, \qquad \text{and} \qquad W_2 = \begin{pmatrix} -L & -L & \cdots & -L \end{pmatrix}, \tag{162}$$

let $\mathcal{W}_1 \in \mathbb{R}^{(2d|\mathcal{M}|)\times d}$, $\mathcal{B}_1 \in \mathbb{R}^{2d|\mathcal{M}|}$, $\mathcal{W}_2 \in \mathbb{R}^{|\mathcal{M}|\times(2d|\mathcal{M}|)}$, $\mathcal{B}_2 \in \mathbb{R}^{|\mathcal{M}|}$ satisfy

$$\mathcal{W}_1 = \begin{pmatrix} W_1 \\ W_1 \\ \vdots \\ W_1 \end{pmatrix}, \quad \mathcal{B}_1 = \begin{pmatrix} B_{m(1)} \\ B_{m(2)} \\ \vdots \\ B_{m(|\mathcal{M}|)} \end{pmatrix}, \quad \mathcal{W}_2 = \begin{pmatrix} W_2 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_2 \end{pmatrix}, \quad \text{and } \mathcal{B}_2 = \begin{pmatrix} f(m(1)) \\ f(m(2)) \\ \vdots \\ f(m(|\mathcal{M}|)) \end{pmatrix}, \tag{163}$$

and let $\Phi \in \mathbf{N}$ satisfy $\Phi = \phi_{|\mathcal{M}|} \bullet ((\mathcal{W}_1, \mathcal{B}_1), (\mathcal{W}_2, \mathcal{B}_2))$ (cf. Definitions 2.9, 2.18, 2.16, and 2.19 and Lemma 3.2). Then

(i) it holds that $\mathcal{D}(\Phi) = (d, 2d|\mathcal{M}|, 2|\mathcal{M}| - 1, 2|\mathcal{M}| - 3, \ldots, 3, 1) \in \mathbb{N}^{|\mathcal{M}|+2}$,

(ii) it holds that $|||\mathcal{T}(\Phi)||| \leq \max\{1, L, \sup_{z\in\mathcal{M}} |||z|||, 2[\sup_{z\in\mathcal{M}} |f(z)|]\}$, and

(iii) it holds that

$$\sup_{x\in D} \big| f(x) - \big(\mathcal{R}_{\mathfrak{r}}(\Phi)\big)(x) \big| \leq 2L \left[\sup_{x=(x_1,x_2,\ldots,x_d)\in D} \left(\inf_{y=(y_1,y_2,\ldots,y_d)\in\mathcal{M}} \sum_{i=1}^d |x_i - y_i| \right) \right] \tag{164}$$

(cf. Definitions 2.11, 2.20, 2.4, and 2.10).

Proof of Proposition 3.5. Throughout this proof let $F \colon \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that

$$F(x) = \max_{y=(y_1,y_2\ldots,y_d)\in\mathcal{M}} \left[f(y) - L\left(\sum_{i=1}^d |x_i - y_i| \right) \right]. \tag{165}$$

Observe that Lemma 3.4 establishes that

(A) it holds that $\mathcal{D}(\Phi) = (d, 2d|\mathcal{M}|, 2|\mathcal{M}| - 1, 2|\mathcal{M}| - 3, \ldots, 3, 1) \in \mathbb{N}^{|\mathcal{M}|+2}$,

(B) it holds that $|||\mathcal{T}(\Phi)||| \leq \max\{1, L, \sup_{z\in\mathcal{M}} |||z|||, 2[\sup_{z\in\mathcal{M}} |f(z)|]\}$, and

(C) it holds for all $x \in D$ that $\big(\mathcal{R}_{\mathfrak{r}}(\Phi)\big)(x) = F(x)$

(cf. Definitions 2.11, 2.20, 2.4, and 2.10). Note that items (A) and (B) prove items (i) and (ii). Next observe that item (C) and Lemma 3.1 (with $E \leftarrow D$, $\delta \leftarrow (D \times D \ni ((x_1, x_2, \dots, x_d), (y_1, y_2, \dots, y_d)) \mapsto \sum_{i=1}^{d} |x_i - y_i| \in [0, \infty))$, $\mathcal{M} \leftarrow \mathcal{M}$, $L \leftarrow L$, $f \leftarrow f$, $F \leftarrow (D \ni x \mapsto F(x) \in \mathbb{R} \cup \{\infty\})$ in the notation of Lemma 3.1) ensure that

$$\begin{aligned} \sup_{x \in D} \big| f(x) - \big(\mathcal{R}_{\mathfrak{r}}(\Phi)\big)(x) \big| &= \sup_{x \in D} |f(x) - F(x)| \\ &\leq 2L \left[\sup_{x=(x_1,x_2,\dots,x_d) \in D} \left(\inf_{y=(y_1,y_2,\dots,y_d) \in \mathcal{M}} \sum_{i=1}^{d} |x_i - y_i| \right) \right]. \end{aligned} \tag{166}$$

The proof of Proposition 3.5 is thus completed. □

3.1.5 Implicit approximations through DNNs

Corollary 3.6. *Let $d, \mathfrak{d} \in \mathbb{N}$, $L \in \mathbb{R}$, let $D \subseteq \mathbb{R}^d$ be a set, let $f \colon D \to \mathbb{R}$ satisfy for all $x = (x_1, x_2, \dots, x_d)$, $y = (y_1, y_2, \dots, y_d) \in D$ that $|f(x) - f(y)| \leq L\big[\sum_{i=1}^{d} |x_i - y_i|\big]$, let $\mathcal{M} \subseteq D$ satisfy $|\mathcal{M}| \in \{2, 3, \dots\}$, and let $l = (l_0, l_1, \dots, l_{|\mathcal{M}|+1}) \in \mathbb{N}^{|\mathcal{M}|+2}$ satisfy $l = (d, 2d|\mathcal{M}|, 2|\mathcal{M}|-1, 2|\mathcal{M}|-3, \dots, 3, 1)$ and $\sum_{k=1}^{|\mathcal{M}|+1} l_k(l_{k-1}+1) \leq \mathfrak{d}$. Then there exists $\theta \in \mathbb{R}^{\mathfrak{d}}$ such that $|||\theta||| \leq \max\{1, L, \sup_{z \in \mathcal{M}} |||z|||, 2[\sup_{z \in \mathcal{M}} |f(z)|]\}$ and*

$$\sup_{x \in D} \big| f(x) - (\mathscr{N}^{\theta,l}_{-\infty,\infty})(x) \big| \leq 2L \left[\sup_{x=(x_1,x_2,\dots,x_d) \in D} \left(\inf_{y=(y_1,y_2,\dots,y_d) \in \mathcal{M}} \sum_{i=1}^{d} |x_i - y_i| \right) \right] \tag{167}$$

(cf. Definitions 2.20 and 2.8).

Proof of Corollary 3.6. Observe that Proposition 3.5 and item (ii) in Lemma 3.2 ensure that there exists $\Phi \in \mathbf{N}$ such that

(A) it holds that $\mathcal{D}(\Phi) = l$,

(B) it holds that $|||\mathcal{T}(\Phi)||| \leq \max\{1, L, \sup_{z \in \mathcal{M}} |||z|||, 2[\sup_{z \in \mathcal{M}} |f(z)|]\}$, and

(C) it holds that

$$\sup_{x \in D} \big| f(x) - \big(\mathcal{R}_{\mathfrak{r}}(\Phi)\big)(x) \big| \leq 2L \left[\sup_{x=(x_1,x_2,\dots,x_d) \in D} \left(\inf_{y=(y_1,y_2,\dots,y_d) \in \mathcal{M}} \sum_{i=1}^{d} |x_i - y_i| \right) \right] \tag{168}$$

(cf. Definitions 2.9, 2.11, 2.20, 2.4, and 2.10). Combining this with Corollary 2.15 establishes (167). The proof of Corollary 3.6 is thus completed. □

Corollary 3.7. *Let $d, \mathfrak{d} \in \mathbb{N}$, $L \in \mathbb{R}$, $u \in [-\infty, \infty)$, $v \in (u, \infty]$, let $D \subseteq \mathbb{R}^d$ be a set, let $f \colon D \to [u, v]$ satisfy for all $x = (x_1, x_2, \dots, x_d)$, $y = (y_1, y_2, \dots, y_d) \in D$ that $|f(x) - f(y)| \leq L\big[\sum_{i=1}^{d} |x_i - y_i|\big]$, let $\mathcal{M} \subseteq D$ satisfy $|\mathcal{M}| \in \{2, 3, \dots\}$, let $l = (l_0, l_1, \dots, l_{|\mathcal{M}|+1}) \in \mathbb{N}^{|\mathcal{M}|+2}$ satisfy $l = (d, 2d|\mathcal{M}|, 2|\mathcal{M}|-1, 2|\mathcal{M}|-3, \dots, 3, 1)$ and $\mathfrak{d} \geq \sum_{k=1}^{|\mathcal{M}|+1} l_k(l_{k-1}+1)$. Then there exists $\theta \in \mathbb{R}^{\mathfrak{d}}$ such that $|||\theta||| \leq \max\{1, L, \sup_{z \in \mathcal{M}} |||z|||, 2[\sup_{z \in \mathcal{M}} |f(z)|]\}$ and*

$$\sup_{x \in D} \big| f(x) - \mathscr{N}^{\theta,l}_{u,v}(x) \big| \leq 2L \left[\sup_{x=(x_1,x_2,\dots,x_d) \in D} \left(\inf_{y=(y_1,y_2,\dots,y_d) \in \mathcal{M}} \sum_{i=1}^{d} |x_i - y_i| \right) \right] \tag{169}$$

(cf. Definitions 2.20 and 2.8).

Proof of Corollary 3.7. First, observe that Corollary 3.6 (with $d \leftarrow d$, $\mathfrak{d} \leftarrow \mathfrak{d}$, $L \leftarrow L$, $D \leftarrow D$, $f \leftarrow (D \ni x \mapsto f(x) \in \mathbb{R})$, $\mathcal{M} \leftarrow \mathcal{M}$, $l \leftarrow l$ in the notation of Corollary 3.6) ensures that there exists $\theta \in \mathbb{R}^{\mathfrak{d}}$ which satisfies $|||\theta||| \leq \max\{1, L, \sup_{z\in\mathcal{M}} |||z|||, 2[\sup_{z\in\mathcal{M}} |f(z)|]\}$ and

$$\sup_{x\in D} \left| f(x) - (\mathscr{N}^{\theta,l}_{-\infty,\infty})(x) \right| \leq 2L \left[\sup_{x=(x_1,x_2,\dots,x_d)\in D} \left(\inf_{y=(y_1,y_2,\dots,y_d)\in\mathcal{M}} \sum_{i=1}^{d} |x_i - y_i| \right) \right] \tag{170}$$

(cf. Definitions 2.20 and 2.8). The assumption that for all $x \in D$ it holds that $u \leq f(x) \leq v$ and Lemma 2.33 hence imply that

$$\begin{aligned} \sup_{x\in D} \left| f(x) - \mathscr{N}^{\theta,l}_{u,v}(x) \right| &= \sup_{x\in D} \left| \mathfrak{C}_{u,v,1}(f(x)) - \mathfrak{C}_{u,v,1}\big((\mathscr{N}^{\theta,l}_{-\infty,\infty})(x)\big) \right| \\ &\leq \sup_{x\in D} \left| f(x) - (\mathscr{N}^{\theta,l}_{-\infty,\infty})(x) \right| \leq 2L \left[\sup_{x=(x_1,x_2,\dots,x_d)\in D} \left(\inf_{y=(y_1,y_2,\dots,y_d)\in\mathcal{M}} \sum_{i=1}^{d} |x_i - y_i| \right) \right] \end{aligned} \tag{171}$$

(cf. Definition 2.7). The proof of Corollary 3.7 is thus completed. □

Corollary 3.8. *Let $d, \mathfrak{d}, \mathfrak{L} \in \mathbb{N}$, $L \in \mathbb{R}$, $u \in [-\infty, \infty)$, $v \in (u, \infty]$, let $D \subseteq \mathbb{R}^d$ be a set, let $f\colon D \to ([u, v] \cap \mathbb{R})$ satisfy for all $x = (x_1, x_2, \dots, x_d)$, $y = (y_1, y_2, \dots, y_d) \in D$ that $|f(x) - f(y)| \leq L\big[\sum_{i=1}^{d} |x_i - y_i|\big]$, let $\mathcal{M} \subseteq D$ satisfy $|\mathcal{M}| \in \{2, 3, \dots\}$, let $l = (l_0, l_1, \dots, l_{\mathfrak{L}}) \in \mathbb{N}^{\mathfrak{L}+1}$ satisfy for all $k \in \{2, 3, \dots, |\mathcal{M}|\}$ that $\mathfrak{L} \geq |\mathcal{M}|+1$, $\sum_{i=1}^{\mathfrak{L}} l_i(l_{i-1}+1) \leq \mathfrak{d}$, $l_0 = d$, $l_{\mathfrak{L}} = 1$, $l_1 \geq 2d|\mathcal{M}|$, and $l_k \geq 2|\mathcal{M}|-2k+3$, and assume for all $i \in \mathbb{N} \cap (|\mathcal{M}|, \mathfrak{L})$ that $l_i \geq 2$. Then there exists $\theta \in \mathbb{R}^{\mathfrak{d}}$ such that $|||\theta||| \leq \max\{1, L, \sup_{z\in\mathcal{M}} |||z|||, 2[\sup_{z\in\mathcal{M}} |f(z)|]\}$ and*

$$\sup_{x\in D} \left| f(x) - \mathscr{N}^{\theta,l}_{u,v}(x) \right| \leq 2L \left[\sup_{x=(x_1,x_2,\dots,x_d)\in D} \left(\inf_{y=(y_1,y_2,\dots,y_d)\in\mathcal{M}} \sum_{i=1}^{d} |x_i - y_i| \right) \right] \tag{172}$$

(cf. Definitions 2.20 and 2.8).

Proof of Corollary 3.8. Throughout this proof let $\mathfrak{l} = (\mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_{|\mathcal{M}|+1}) \in \mathbb{N}^{|\mathcal{M}|+2}$ satisfy $\mathfrak{l} = (d, 2d|\mathcal{M}|, 2|\mathcal{M}|-1, 2|\mathcal{M}|-3, \dots, 3, 1)$. First, note that Corollary 3.7 (with $d \leftarrow d$, $\mathfrak{d} \leftarrow \sum_{k=1}^{|\mathcal{M}|+1} \mathfrak{l}_k(\mathfrak{l}_{k-1}+1)$, $L \leftarrow L$, $u \leftarrow u$, $v \leftarrow v$, $D \leftarrow D$, $f \leftarrow f$, $\mathcal{M} \leftarrow \mathcal{M}$, $l \leftarrow \mathfrak{l}$ in the notation of Corollary 3.7) establishes that there exists $\eta \in \mathbb{R}^{\sum_{k=1}^{|\mathcal{M}|+1} \mathfrak{l}_k(\mathfrak{l}_{k-1}+1)}$ which satisfies $|||\eta||| \leq \max\{1, L, \sup_{z\in\mathcal{M}} |||z|||, 2[\sup_{z\in\mathcal{M}} |f(z)|]\}$ and

$$\sup_{x\in D} \left| f(x) - \mathscr{N}^{\eta,\mathfrak{l}}_{u,v}(x) \right| \leq 2L \left[\sup_{x=(x_1,x_2,\dots,x_d)\in D} \left(\inf_{y=(y_1,y_2,\dots,y_d)\in\mathcal{M}} \sum_{i=1}^{d} |x_i - y_i| \right) \right] \tag{173}$$

(cf. Definitions 2.20 and 2.8). Next observe that Lemma 2.30 (with $u \leftarrow u$, $v \leftarrow v$, $L \leftarrow |\mathcal{M}|+1$, $\mathfrak{L} \leftarrow \mathfrak{L}$, $d \leftarrow \sum_{k=1}^{|\mathcal{M}|+1} \mathfrak{l}_k(\mathfrak{l}_{k-1}+1)$, $\mathfrak{d} \leftarrow \mathfrak{d}$, $\theta \leftarrow \eta$, $(l_0, l_1, \dots, l_L) \leftarrow (\mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_{|\mathcal{M}|+1})$, $(\mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_{\mathfrak{L}}) \leftarrow (l_0, l_1, \dots, l_{\mathfrak{L}})$, in the notation of Lemma 2.30) shows that there exists $\theta \in \mathbb{R}^d$ such that

$$|||\theta||| \leq \max\{1, |||\eta|||\} \qquad \text{and} \qquad \mathscr{N}^{\theta,l}_{u,v} = \mathscr{N}^{\eta,\mathfrak{l}}_{u,v}. \tag{174}$$

Combining this with (173) proves (172). The proof of Corollary 3.8 is thus completed. □

Corollary 3.9. *Let $d, \mathfrak{d}, N \in \mathbb{N}$, $L \in \mathbb{R}$, $u \in [-\infty, \infty)$, $v \in (u, \infty]$ satisfy $\mathfrak{d} \geq 2d^2(N+1)^d + 5d(N+1)^{2d} + \frac{4}{3}(N+1)^{3d}$, let $\|\cdot\|\colon \mathbb{R}^d \to [0, \infty)$ be the standard norm, let $p = (p_1, p_2, \dots, p_d)$, $q = (q_1, q_2, \dots, q_d) \in \mathbb{R}^d$ satisfy for all $i \in \{1, 2, \dots, d\}$ that $p_i \leq q_i$ and $\max_{j\in\{1,2,\dots,d\}}(q_j - p_j) > 0$, let $D = \prod_{i=1}^{d} [p_i, q_i]$, let $\mathcal{M} \subseteq D$ satisfy*

$$\mathcal{M} = \left\{ y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d \colon \begin{pmatrix} \exists\, k_1, k_2, \dots, k_d \in \{0, 1, \dots, N\}\colon \\ \forall\, i \in \{1, 2, \dots, d\}\colon y_i = p_i + \frac{k_i}{N}(q_i - p_i) \end{pmatrix} \right\}, \tag{175}$$

and let $f\colon D\to([u,v]\cap\mathbb{R})$ *satisfy for all* $x,y\in D$ *that* $|f(x)-f(y)|\le L\|x-y\|$*. Then there exist* $\theta\in\mathbb{R}^{\mathfrak{d}}$*,* $\mathfrak{L}\in\mathbb{N}$*,* $l=(l_0,l_1,\dots,l_{\mathfrak{L}})\in\mathbb{N}^{\mathfrak{L}+1}$ *such that* $|||\theta|||\le\max\{1,L,|||p|||,|||q|||,2[\sup_{z\in D}|f(z)|]\}$*,* $\sum_{k=1}^{\mathfrak{L}}l_k(l_{k-1}+1)\le\mathfrak{d}$*, and*

$$\sup_{x\in D}\left|f(x)-\mathscr{N}^{\theta,l}_{u,v}(x)\right|\le\frac{L}{N}\left[\sum_{i=1}^{d}|q_i-p_i|\right]\tag{176}$$

(cf. Definitions 2.20 and 2.8).

Proof of Corollary 3.9. Throughout this proof let $l=(l_0,l_1,\dots,l_{|\mathcal{M}|+1})\in\mathbb{N}^{|\mathcal{M}|+2}$ satisfy $l=(d,2d|\mathcal{M}|,2|\mathcal{M}|-1,2|\mathcal{M}|-3,\dots,3,1)$. Observe that the fact that $|\mathcal{M}|\le(N+1)^d$, the fact that for all $n\in\mathbb{N}$ it holds that $\sum_{i=1}^{n}(2i-1)=n^2$, and the fact that for all $n\in\mathbb{N}$ it holds that $\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}\le\frac{(n+1)^3}{3}$ ensure that

$$\begin{aligned}
&\sum_{k=1}^{|\mathcal{M}|+1}l_k(l_{k-1}+1)\\
&=\underbrace{d(2d|\mathcal{M}|)+2d|\mathcal{M}|(2|\mathcal{M}|-1)+\left[\sum_{i=1}^{|\mathcal{M}|-1}(2i+1)(2i-1)\right]}_{\text{number of weights}}+\underbrace{2d|\mathcal{M}|+\left[\sum_{i=1}^{|\mathcal{M}|}(2i-1)\right]}_{\text{number of biases}}\\
&=2d^2|\mathcal{M}|+4d|\mathcal{M}|^2-2d|\mathcal{M}|+\left[\sum_{i=1}^{|\mathcal{M}|-1}(4i^2-1)\right]+2d|\mathcal{M}|+|\mathcal{M}|^2\\
&=2d^2|\mathcal{M}|+(4d+1)|\mathcal{M}|^2+4\left[\sum_{i=1}^{|\mathcal{M}|-1}i^2\right]-(|\mathcal{M}|-1)\\
&\le2d^2|\mathcal{M}|+5d|\mathcal{M}|^2+\tfrac{4}{3}|\mathcal{M}|^3\le2d^2(N+1)^d+5d(N+1)^{2d}+\tfrac{4}{3}(N+1)^{3d}\le\mathfrak{d}.
\end{aligned}\tag{177}$$

In addition, note that the hypothesis that for all $x,y\in D$ it holds that $|f(x)-f(y)|\le L\|x-y\|$ implies that for all $x=(x_1,x_2,\dots,x_d)$, $y=(y_1,y_2,\dots,y_d)\in D$ it holds that

$$|f(x)-f(y)|\le L\left[\sum_{i=1}^{d}|x_i-y_i|\right].\tag{178}$$

Furthermore, observe that the hypothesis that $\max_{j\in\{1,2,\dots,d\}}(q_j-p_j)>0$ ensures that $|\mathcal{M}|\ge2$. Combining this, (177), and (178) with Corollary 3.7 establishes that there exists $\theta\in\mathbb{R}^{\mathfrak{d}}$ such that $|||\theta|||\le\max\{1,L,\sup_{z\in\mathcal{M}}|||z|||,2[\sup_{z\in\mathcal{M}}|f(z)|]\}$ and

$$\sup_{x\in D}\left|f(x)-\mathscr{N}^{\theta,l}_{u,v}(x)\right|\le2L\left[\sup_{x=(x_1,x_2,\dots,x_d)\in D}\left(\inf_{y=(y_1,y_2,\dots,y_d)\in\mathcal{M}}\sum_{i=1}^{d}|x_i-y_i|\right)\right]\tag{179}$$

(cf. Definitions 2.20 and 2.8). Next note that the hypothesis that $\mathcal{M}\subseteq D=\prod_{i=1}^{d}[p_i,q_i]$ implies that for all $z\in\mathcal{M}$ it holds that

$$|||z|||\le\max\{|||p|||,|||q|||\}.\tag{180}$$

Therefore, we obtain that

$$|||\theta|||\le\max\left\{1,L,|||p|||,|||q|||,2\left[\sup_{z\in\mathcal{M}}|f(z)|\right]\right\}\le\max\left\{1,L,|||p|||,|||q|||,2\left[\sup_{z\in D}|f(z)|\right]\right\}.\tag{181}$$

In the next step we note that the fact that for all $N\in\mathbb{N}$, $r\in\mathbb{R}$, $s\in[r,\infty)$, $x\in[r,s]$ there exists $k\in\{0,1,\dots,N\}$ such that $|x-(r+\frac{k}{N}(s-r))|\le\frac{s-r}{2N}$ ensures that for all $x=(x_1,x_2,\dots,x_d)\in D$ there exists $y=(y_1,y_2,\dots,y_d)\in\mathcal{M}$ such that

$$\sum_{i=1}^{d}|x_i-y_i|\le\frac{1}{2N}\left[\sum_{i=1}^{d}|q_i-p_i|\right].\tag{182}$$

Combining this, (177), (179), and (181) establishes (176). The proof of Corollary 3.9 is thus completed. ☐

3.2 Analysis of the generalization error

3.2.1 Hoeffding's concentration inequality

Proposition 3.10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $N \in \mathbb{N}$, $\varepsilon \in [0, \infty)$, $a_1, a_2, \ldots, a_N \in \mathbb{R}$, $b_1 \in [a_1, \infty)$, $b_2 \in [a_2, \infty)$, $\ldots$, $b_N \in [a_N, \infty)$, assume $\sum_{n=1}^{N}(b_n - a_n)^2 \neq 0$, and let $X_n \colon \Omega \to [a_n, b_n]$, $n \in \{1, 2, \ldots, N\}$, be independent random variables. Then*

$$\mathbb{P}\left(\frac{1}{N}\left|\sum_{n=1}^{N}\big(X_n - \mathbb{E}[X_n]\big)\right| \geq \varepsilon\right) \leq 2\exp\left(\frac{-2\varepsilon^2 N^2}{\sum_{n=1}^{N}(b_n - a_n)^2}\right). \tag{183}$$

3.2.2 Covering number estimates

Definition 3.11 (Covering number)**.** Let (E, δ) be a metric space and let $r \in [0, \infty]$. Then we denote by $\mathcal{C}_{(E,\delta),r} \in \mathbb{N}_0 \cup \{\infty\}$ (we denote by $\mathcal{C}_{E,r} \in \mathbb{N}_0 \cup \{\infty\}$) the extended real number given by

$$\mathcal{C}_{(E,\delta),r} = \inf\Big(\Big\{n \in \mathbb{N}_0 \colon \big(\exists\, A \subseteq E \colon \big[(|A| \leq n) \wedge (\forall\, x \in E \colon \exists\, a \in A \colon \delta(a, x) \leq r)\big]\big)\Big\} \cup \{\infty\}\Big). \tag{184}$$

Proposition 3.12. *Let $(X, \|\cdot\|)$ be a finite-dimensional Banach space, let $R, r \in (0, \infty)$, $B = \{\theta \in X \colon \|\theta\| \leq R\}$, and let $\delta \colon B \times B \to [0, \infty)$ satisfy for all $\theta, \vartheta \in B$ that $\delta(\theta, \vartheta) = \|\theta - \vartheta\|$. Then*

$$\mathcal{C}_{(B,\delta),r} \leq \begin{cases} 1 & : r \geq R \\ \left\lceil \frac{4R}{r} \right\rceil^{\dim(X)} & : r < R \end{cases} \tag{185}$$

(cf. Definition 3.11).

3.2.3 Measurability properties for suprema

Lemma 3.13. *Let $(E, \mathscr{E})$ be a topological space, assume $E \neq \emptyset$, let $\mathbf{E} \subseteq E$ be an at most countable set, assume that $\mathbf{E}$ is dense in E, let $(\Omega, \mathcal{F})$ be a measurable space, let $f_x \colon \Omega \to \mathbb{R}$, $x \in E$, be $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable functions, assume for all $\omega \in \Omega$ that $E \ni x \mapsto f_x(\omega) \in \mathbb{R}$ is a continuous function, and let $F \colon \Omega \to \mathbb{R} \cup \{\infty\}$ satisfy for all $\omega \in \Omega$ that $F(\omega) = \sup_{x \in E} f_x(\omega)$. Then*

(i) it holds for all $\omega \in \Omega$ that $F(\omega) = \sup_{x \in \mathbf{E}} f_x(\omega)$ and

(ii) it holds that F is an $\mathcal{F}/\mathcal{B}(\mathbb{R} \cup \{\infty\})$-measurable function.

Proof of Lemma 3.13. Note that the hypothesis that $\mathbf{E}$ is dense in E implies that for all $g \in C(E, \mathbb{R})$ it holds that

$$\sup_{x \in E} g(x) = \sup_{x \in \mathbf{E}} g(x). \tag{186}$$

This and the hypothesis that for all $\omega \in \Omega$ it holds that $E \ni x \mapsto f_x(\omega) \in \mathbb{R}$ is a continuous function show that for all $\omega \in \Omega$ it holds that

$$F(\omega) = \sup_{x \in E} f_x(\omega) = \sup_{x \in \mathbf{E}} f_x(\omega). \tag{187}$$

This establishes item (i). Next note that item (i) and the hypothesis that for all $x \in E$ it holds that $f_x \colon \Omega \to \mathbb{R}$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function demonstrate item (ii). The proof of Lemma 3.13 is thus completed. ∎

Lemma 3.14. *Let (E, δ) be a separable metric space, assume $E \neq \emptyset$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $L \in \mathbb{R}$, and let $Z_x \colon \Omega \to \mathbb{R}$, $x \in E$, be random variables which satisfy for all $x, y \in E$ that $\mathbb{E}[|Z_x|] < \infty$ and $|Z_x - Z_y| \leq L\delta(x, y)$. Then*

(i) *it holds for all* $x, y \in E$, $\eta \in \Omega$ *that* $|(Z_x(\eta) - \mathbb{E}[Z_x]) - (Z_y(\eta) - \mathbb{E}[Z_y])| \leq 2L\delta(x, y)$ *and*

(ii) *it holds that* $\Omega \ni \eta \mapsto \sup_{x \in E}|Z_x(\eta) - \mathbb{E}[Z_x]| \in [0, \infty]$ *is an* $\mathcal{F}/\mathcal{B}([0, \infty])$*-measurable function.*

Proof of Lemma 3.14. Note that the hypothesis that for all $x, y \in E$ it holds that $|Z_x - Z_y| \leq L\delta(x, y)$ shows that for all $x, y \in E$, $\eta \in \Omega$ it holds that

$$\begin{aligned}
&|(Z_x(\eta) - \mathbb{E}[Z_x]) - (Z_y(\eta) - \mathbb{E}[Z_y])| = |(Z_x(\eta) - Z_y(\eta)) + (\mathbb{E}[Z_y] - \mathbb{E}[Z_x])| \\
&\leq |Z_x(\eta) - Z_y(\eta)| + |\mathbb{E}[Z_x] - \mathbb{E}[Z_y]| \leq L\delta(x, y) + |\mathbb{E}[Z_x] - \mathbb{E}[Z_y]| \\
&= L\delta(x, y) + |\mathbb{E}[Z_x - Z_y]| \leq L\delta(x, y) + \mathbb{E}[|Z_x - Z_y|] \leq L\delta(x, y) + L\delta(x, y) = 2L\delta(x, y).
\end{aligned} \tag{188}$$

This proves item (i). Next observe that item (i) implies that for all $\eta \in \Omega$ it holds that $E \ni x \mapsto |Z_x(\eta) - \mathbb{E}[Z_x]| \in \mathbb{R}$ is a continuous function. Combining this and the hypothesis that E is separable with Lemma 3.13 establishes item (ii). The proof of Lemma 3.14 is thus completed. □

3.2.4 Concentration inequalities for random fields

Lemma 3.15. *Let* (E, δ) *be a separable metric space, let* $\varepsilon, L \in \mathbb{R}$, $N \in \mathbb{N}$, $z_1, z_2, \ldots, z_N \in E$ *satisfy* $E \subseteq \bigcup_{i=1}^N \{x \in E \colon 2L\delta(x, z_i) \leq \varepsilon\}$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, and let* $Z_x \colon \Omega \to \mathbb{R}$, $x \in E$, *be random variables which satisfy for all* $x, y \in E$ *that* $|Z_x - Z_y| \leq L\delta(x, y)$. *Then*

$$\mathbb{P}(\sup_{x \in E}|Z_x| \geq \varepsilon) \leq \sum_{i=1}^N \mathbb{P}\big(|Z_{z_i}| \geq \tfrac{\varepsilon}{2}\big) \tag{189}$$

(cf. Lemma 3.13).

Proof of Lemma 3.15. Throughout this proof let $B_1, B_2, \ldots, B_N \subseteq E$ satisfy for all $i \in \{1, 2, \ldots, N\}$ that $B_i = \{x \in E \colon 2L\delta(x, z_i) \leq \varepsilon\}$. Observe that the triangle inequality and the hypothesis that for all $x, y \in E$ it holds that $|Z_x - Z_y| \leq L\delta(x, y)$ show that for all $i \in \{1, 2, \ldots, N\}$, $x \in B_i$ it holds that

$$|Z_x| = |Z_x - Z_{z_i} + Z_{z_i}| \leq |Z_x - Z_{z_i}| + |Z_{z_i}| \leq L\delta(x, z_i) + |Z_{z_i}| \leq \tfrac{\varepsilon}{2} + |Z_{z_i}|. \tag{190}$$

Combining this with Lemma 3.13 proves that for all $i \in \{1, 2, \ldots, N\}$ it holds that

$$\mathbb{P}\big(\sup_{x \in B_i}|Z_x| \geq \varepsilon\big) \leq \mathbb{P}\big(\tfrac{\varepsilon}{2} + |Z_{z_i}| \geq \varepsilon\big) = \mathbb{P}\big(|Z_{z_i}| \geq \tfrac{\varepsilon}{2}\big). \tag{191}$$

This and Lemma 3.13 establish that

$$\begin{aligned}
\mathbb{P}(\sup_{x \in E}|Z_x| \geq \varepsilon) &= \mathbb{P}\Big(\sup_{x \in (\bigcup_{i=1}^N B_i)}|Z_x| \geq \varepsilon\Big) = \mathbb{P}\Big(\textstyle\bigcup_{i=1}^N \big\{\sup_{x \in B_i}|Z_x| \geq \varepsilon\big\}\Big) \\
&\leq \sum_{i=1}^N \mathbb{P}\big(\sup_{x \in B_i}|Z_x| \geq \varepsilon\big) \leq \sum_{i=1}^N \mathbb{P}\big(|Z_{z_i}| \geq \tfrac{\varepsilon}{2}\big).
\end{aligned} \tag{192}$$

This completes the proof of Lemma 3.15. □

Lemma 3.16. *Let* (E, δ) *be a separable metric space, assume* $E \neq \emptyset$, *let* $\varepsilon, L \in (0, \infty)$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, and let* $Z_x \colon \Omega \to \mathbb{R}$, $x \in E$, *be random variables which satisfy for all* $x, y \in E$ *that* $|Z_x - Z_y| \leq L\delta(x, y)$. *Then*

$$\big[\mathcal{C}_{(E,\delta), \frac{\varepsilon}{2L}}\big]^{-1} \mathbb{P}(\sup_{x \in E}|Z_x| \geq \varepsilon) \leq \sup_{x \in E} \mathbb{P}\big(|Z_x| \geq \tfrac{\varepsilon}{2}\big). \tag{193}$$

(cf. Definition 3.11 and Lemma 3.13).

Proof of Lemma 3.16. Throughout this proof let $N \in \mathbb{N} \cup \{\infty\}$ satisfy $N = \mathcal{C}_{(E,\delta),\frac{\varepsilon}{2L}}$, assume without loss of generality that $N < \infty$, and let $z_1, z_2, \ldots, z_N \in E$ satisfy $E \subseteq \bigcup_{i=1}^N \{x \in E \colon \delta(x, z_i) \leq \frac{\varepsilon}{2L}\}$ (cf. Definition 3.11). Observe that Lemma 3.13 and Lemma 3.15 establish that

$$\mathbb{P}(\sup_{x\in E}|Z_x| \geq \varepsilon) \leq \sum_{i=1}^{N} \mathbb{P}\big(|Z_{z_i}| \geq \tfrac{\varepsilon}{2}\big) \leq N\big[\sup_{x\in E} \mathbb{P}\big(|Z_x| \geq \tfrac{\varepsilon}{2}\big)\big]. \tag{194}$$

This completes the proof of Lemma 3.16. ☐

Lemma 3.17. *Let (E, δ) be a separable metric space, assume $E \neq \emptyset$, let $\varepsilon, L \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $Z_x \colon \Omega \to \mathbb{R}$, $x \in E$, be random variables which satisfy for all $x, y \in E$ that $\mathbb{E}[|Z_x|] < \infty$ and $|Z_x - Z_y| \leq L\delta(x, y)$. Then*

$$\big[\mathcal{C}_{(E,\delta),\frac{\varepsilon}{4L}}\big]^{-1}\mathbb{P}(\sup_{x\in E}|Z_x - \mathbb{E}[Z_x]| \geq \varepsilon) \leq \sup_{x\in E} \mathbb{P}\big(|Z_x - \mathbb{E}[Z_x]| \geq \tfrac{\varepsilon}{2}\big). \tag{195}$$

(cf. Definition 3.11 and Lemma 3.14).

Proof of Lemma 3.17. Throughout this proof let $Y_x \colon \Omega \to \mathbb{R}$, $x \in E$, satisfy for all $x \in E$, $\eta \in \Omega$ that $Y_x(\eta) = Z_x(\eta) - \mathbb{E}[Z_x]$. Observe that Lemma 3.14 ensures that for all $x, y \in E$ it holds that

$$|Y_x - Y_y| \leq 2L\delta(x, y). \tag{196}$$

This and Lemma 3.16 (with $(E, \delta) \leftarrow (E, \delta)$, $\varepsilon \leftarrow \varepsilon$, $L \leftarrow 2L$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(Z_x)_{x\in E} \leftarrow (Y_x)_{x\in E}$ in the notation of Lemma 3.16) establish (195). The proof of Lemma 3.17 is thus completed. ☐

Lemma 3.18. *Let (E, δ) be a separable metric space, assume $E \neq \emptyset$, let $M \in \mathbb{N}$, $\varepsilon, L, D \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every $x \in E$ let $Y_{x,1}, Y_{x,2}, \ldots, Y_{x,M} \colon \Omega \to [0, D]$ be independent random variables, assume for all $x, y \in E$, $m \in \{1, 2, \ldots, M\}$ that $|Y_{x,m} - Y_{y,m}| \leq L\delta(x, y)$, and let $Z_x \colon \Omega \to [0, \infty)$, $x \in E$, satisfy for all $x \in E$ that*

$$Z_x = \frac{1}{M}\left[\sum_{m=1}^{M} Y_{x,m}\right]. \tag{197}$$

Then

(i) it holds for all $x \in E$ that $\mathbb{E}[|Z_x|] \leq D < \infty$,

(ii) it holds that $\Omega \ni \eta \mapsto \sup_{x\in E}|Z_x(\eta) - \mathbb{E}[Z_x]| \in [0, \infty]$ is an $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable function, and

(iii) it holds that

$$\mathbb{P}(\sup_{x\in E}|Z_x - \mathbb{E}[Z_x]| \geq \varepsilon) \leq 2\mathcal{C}_{(E,\delta),\frac{\varepsilon}{4L}} \exp\!\left(\frac{-\varepsilon^2 M}{2D^2}\right) \tag{198}$$

(cf. Definition 3.11).

Proof of Lemma 3.18. First, observe that the triangle inequality and the hypothesis that for all $x, y \in E$, $m \in \{1, 2, \ldots, M\}$ it holds that $|Y_{x,m} - Y_{y,m}| \leq L\delta(x, y)$ imply that for all $x, y \in E$ it holds that

$$\begin{aligned} |Z_x - Z_y| &= \left|\frac{1}{M}\left[\sum_{m=1}^{M} Y_{x,m}\right] - \frac{1}{M}\left[\sum_{m=1}^{M} Y_{y,m}\right]\right| = \frac{1}{M}\left|\sum_{m=1}^{M}\big(Y_{x,m} - Y_{y,m}\big)\right| \\ &\leq \frac{1}{M}\left[\sum_{m=1}^{M}\big|Y_{x,m} - Y_{y,m}\big|\right] \leq L\delta(x, y). \end{aligned} \tag{199}$$

Next note that the hypothesis that for all $x \in E$, $m \in \{1, 2, \dots, M\}$, $\omega \in \Omega$ it holds that $|Y_{x,m}(\omega)| \in [0, D]$ ensures that for all $x \in E$ it holds that

$$\mathbb{E}\big[|Z_x|\big] = \mathbb{E}\left[\frac{1}{M}\left[\sum_{m=1}^{M} Y_{x,m}\right]\right] = \frac{1}{M}\left[\sum_{m=1}^{M} \mathbb{E}\big[Y_{x,m}\big]\right] \leq D < \infty. \tag{200}$$

This proves item (i). Furthermore, note that item (i), (199), and Lemma 3.14 establish item (ii). Next observe that (197) shows that for all $x \in E$ it holds that

$$|Z_x - \mathbb{E}[Z_x]| = \left|\frac{1}{M}\left[\sum_{m=1}^{M} Y_{x,m}\right] - \mathbb{E}\left[\frac{1}{M}\left[\sum_{m=1}^{M} Y_{x,m}\right]\right]\right| = \frac{1}{M}\left|\sum_{m=1}^{M}\big(Y_{x,m} - \mathbb{E}\big[Y_{x,m}\big]\big)\right|. \tag{201}$$

Combining this with Proposition 3.10 (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $N \leftarrow M$, $\varepsilon \leftarrow \frac{\varepsilon}{2}$, $(a_1, a_2, \dots, a_N) \leftarrow (0, 0, \dots, 0)$, $(b_1, b_2, \dots, b_N) \leftarrow (D, D, \dots, D)$, $(X_n)_{n \in \{1,2,\dots,N\}} \leftarrow (Y_{x,m})_{m \in \{1,2,\dots,M\}}$ for $x \in E$ in the notation of Proposition 3.10) ensures that for all $x \in E$ it holds that

$$\mathbb{P}\big(|Z_x - \mathbb{E}[Z_x]| \geq \tfrac{\varepsilon}{2}\big) \leq 2\exp\left(\frac{-2\big[\frac{\varepsilon}{2}\big]^2 M^2}{MD^2}\right) = 2\exp\left(\frac{-\varepsilon^2 M}{2D^2}\right). \tag{202}$$

Combining this, (199), and (200) with Lemma 3.17 establishes item (iii). The proof of Lemma 3.18 is thus completed. □

3.2.5 Uniform estimates for the statistical learning error

Lemma 3.19. *Let (E, δ) be a separable metric space, assume $E \neq \emptyset$, let $M \in \mathbb{N}$, $\varepsilon, L, D \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X_{x,m} \colon \Omega \to \mathbb{R}$, $x \in E$, $m \in \{1, 2, \dots, M\}$, and $Y_m \colon \Omega \to \mathbb{R}$, $m \in \{1, 2, \dots, M\}$, be functions, assume for all $x \in E$ that $(X_{x,m}, Y_m)$, $m \in \{1, 2, \dots, M\}$, are i.i.d. random variables, assume for all $x, y \in E$, $m \in \{1, 2, \dots, M\}$ that $|X_{x,m} - X_{y,m}| \leq L\delta(x, y)$ and $|X_{x,m} - Y_m| \leq D$, let $\mathfrak{E}_x \colon \Omega \to [0, \infty)$, $x \in E$, satisfy for all $x \in E$ that*

$$\mathfrak{E}_x = \frac{1}{M}\left[\sum_{m=1}^{M} |X_{x,m} - Y_m|^2\right], \tag{203}$$

and let $\mathcal{E}_x \in [0, \infty)$, $x \in E$, satisfy for all $x \in E$ that $\mathcal{E}_x = \mathbb{E}[|X_{x,1} - Y_1|^2]$. Then $\Omega \ni \omega \mapsto \sup_{x \in E}|\mathfrak{E}_x(\omega) - \mathcal{E}_x| \in [0, \infty]$ is an $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable function and

$$\mathbb{P}(\sup_{x \in E}|\mathfrak{E}_x - \mathcal{E}_x| \geq \varepsilon) \leq 2\mathcal{C}_{(E,\delta), \frac{\varepsilon}{8LD}} \exp\left(\frac{-\varepsilon^2 M}{2D^4}\right) \tag{204}$$

(cf. Definition 3.11).

Proof of Lemma 3.19. Throughout this proof let $\mathscr{E}_{x,m} \colon \Omega \to [0, D^2]$, $x \in E$, $m \in \{1, 2, \dots, M\}$, satisfy for all $x \in E$, $m \in \{1, 2, \dots, M\}$ that

$$\mathscr{E}_{x,m} = |X_{x,m} - Y_m|^2. \tag{205}$$

Observe that the fact that for all $x_1, x_2, y \in \mathbb{R}$ it holds that $(x_1 - y)^2 - (x_2 - y)^2 = (x_1 - x_2)((x_1 - y) + (x_2 - y))$, the hypothesis that for all $x \in E$, $m \in \{1, 2, \dots, M\}$ it holds that $|X_{x,m} - Y_m| \leq D$, and the hypothesis that for all $x, y \in E$, $m \in \{1, 2, \dots, M\}$ it holds that $|X_{x,m} - X_{y,m}| \leq L\delta(x, y)$ imply that for all $x, y \in E$, $m \in \{1, 2, \dots, M\}$ it holds that

$$\begin{aligned} |\mathscr{E}_{x,m} - \mathscr{E}_{y,m}| &= \big|(X_{x,m} - Y_m)^2 - (X_{y,m} - Y_m)^2\big| = |X_{x,m} - X_{y,m}|\big|(X_{x,m} - Y_m) + (X_{y,m} - Y_m)\big| \\ &\leq |X_{x,m} - X_{y,m}|\big(|X_{x,m} - Y_m| + |X_{y,m} - Y_m|\big) \leq 2D|X_{x,m} - X_{y,m}| \leq 2LD\delta(x, y). \end{aligned} \tag{206}$$

In addition, note that (203) and the hypothesis that for all $x \in E$ it holds that $(X_{x,m}, Y_m)$, $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables show that for all $x \in E$ it holds that

$$\mathbb{E}\big[\mathfrak{E}_x\big] = \frac{1}{M}\left[\sum_{m=1}^{M} \mathbb{E}\big[|X_{x,m} - Y_m|^2\big]\right] = \frac{1}{M}\left[\sum_{m=1}^{M} \mathbb{E}\big[|X_{x,1} - Y_1|^2\big]\right] = \frac{1}{M}\left[\sum_{m=1}^{M} \mathcal{E}_x\right] = \mathcal{E}_x. \tag{207}$$

Furthermore, observe that the hypothesis that for all $x \in E$ it holds that $(X_{x,m}, Y_m)$, $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables ensures that for all $x \in E$ it holds that $\mathscr{E}_{x,m}$, $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables. Combining this, (206), and (207) with Lemma 3.18 (with $(E, \delta) \leftarrow (E, \delta)$, $M \leftarrow M$, $\varepsilon \leftarrow \varepsilon$, $L \leftarrow 2LD$, $D \leftarrow D^2$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(Y_{x,m})_{x\in E,\, m\in\{1,2,\ldots,M\}} \leftarrow (\mathscr{E}_{x,m})_{x\in E,\, m\in\{1,2,\ldots,M\}}$, $(Z_x)_{x\in E} = (\mathfrak{E}_x)_{x\in E}$ in the notation of Lemma 3.18) establishes (204). The proof of Lemma 3.19 is thus completed. □

Lemma 3.20. *Let $d, \mathfrak{d}, M \in \mathbb{N}$, $R, L, \mathcal{R}, \varepsilon \in (0, \infty)$, let $D \subseteq \mathbb{R}^d$ be a compact set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X_m \colon \Omega \to D$, $m \in \{1, 2, \ldots, M\}$, and $Y_m \colon \Omega \to \mathbb{R}$, $m \in \{1, 2, \ldots, M\}$, be functions, assume that (X_m, Y_m), $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables, let $H = (H_\theta)_{\theta\in[-R,R]^{\mathfrak{d}}} \colon [-R, R]^{\mathfrak{d}} \to C(D, \mathbb{R})$ satisfy for all $\theta, \vartheta \in [-R, R]^{\mathfrak{d}}$, $x \in D$ that $|H_\theta(x) - H_\vartheta(x)| \leq L|||\theta - \vartheta|||$, assume for all $\theta \in [-R, R]^{\mathfrak{d}}$, $m \in \{1, 2, \ldots, M\}$ that $|H_\theta(X_m) - Y_m| \leq \mathcal{R}$ and $\mathbb{E}[|Y_1|^2] < \infty$, let $\mathcal{E} \colon C(D, \mathbb{R}) \to [0, \infty)$ satisfy for all $f \in C(D, \mathbb{R})$ that $\mathcal{E}(f) = \mathbb{E}[|f(X_1) - Y_1|^2]$, and let $\mathfrak{E} \colon [-R, R]^{\mathfrak{d}} \times \Omega \to [0, \infty)$ satisfy for all $\theta \in [-R, R]^{\mathfrak{d}}$, $\omega \in \Omega$ that*

$$\mathfrak{E}(\theta, \omega) = \frac{1}{M}\left[\sum_{m=1}^{M} |H_\theta(X_m(\omega)) - Y_m(\omega)|^2\right] \tag{208}$$

(cf. Definition 2.20). Then $\Omega \ni \omega \mapsto \sup_{\theta\in[-R,R]^{\mathfrak{d}}} |\mathfrak{E}(\theta, \omega) - \mathcal{E}(H_\theta)| \in [0, \infty]$ is an $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable function and

$$\mathbb{P}\big(\sup\nolimits_{\theta\in[-R,R]^{\mathfrak{d}}} |\mathfrak{E}(\theta) - \mathcal{E}(H_\theta)| \geq \varepsilon\big) \leq 2 \max\left\{1, \left[\frac{32LR\mathcal{R}}{\varepsilon}\right]^{\mathfrak{d}}\right\} \exp\left(\frac{-\varepsilon^2 M}{2\mathcal{R}^4}\right). \tag{209}$$

Proof of Lemma 3.20. Throughout this proof let $B \subseteq \mathbb{R}^{\mathfrak{d}}$ satisfy $B = [-R, R]^{\mathfrak{d}} = \{\theta \in \mathbb{R}^{\mathfrak{d}} \colon |||\theta||| \leq R\}$ and let $\delta \colon B \times B \to [0, \infty)$ satisfy for all $\theta, \vartheta \in B$ that

$$\delta(\theta, \vartheta) = |||\theta - \vartheta|||. \tag{210}$$

Observe that the hypothesis that (X_m, Y_m), $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables and the hypothesis that for all $\theta \in [-R, R]^{\mathfrak{d}}$ it holds that H_θ is a continuous function imply that for all $\theta \in B$ it holds that $(H_\theta(X_m), Y_m)$, $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables. Combining this, the hypothesis that for all $\theta, \vartheta \in B$, $x \in D$ it holds that $|H_\theta(x) - H_\vartheta(x)| \leq L|||\theta - \vartheta|||$, and the hypothesis that for all $\theta \in B$, $m \in \{1, 2, \ldots, M\}$ it holds that $|H_\theta(X_m) - Y_m| \leq \mathcal{R}$ with Lemma 3.19 (with $(E, \delta) \leftarrow (B, \delta)$, $M \leftarrow M$, $\varepsilon \leftarrow \varepsilon$, $L \leftarrow L$, $D \leftarrow \mathcal{R}$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(X_{x,m})_{x\in E,\, m\in\{1,2,\ldots,M\}} \leftarrow (H_\theta(X_m))_{\theta\in B,\, m\in\{1,2,\ldots,M\}}$, $(Y_m)_{m\in\{1,2,\ldots,M\}} \leftarrow (Y_m)_{m\in\{1,2,\ldots,M\}}$, $(\mathfrak{E}_x)_{x\in E} \leftarrow \big((\Omega \ni \omega \mapsto \mathfrak{E}(\theta, \omega) \in [0, \infty))\big)_{\theta\in B}$, $(\mathcal{E}_x)_{x\in E} \leftarrow (\mathcal{E}(H_\theta))_{\theta\in B}$ in the notation of Lemma 3.19) establishes that $\Omega \ni \omega \mapsto \sup_{\theta\in B} |\mathfrak{E}(\theta, \omega) - \mathcal{E}(H_\theta)| \in [0, \infty]$ is an $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable function and

$$\mathbb{P}\big(\sup\nolimits_{\theta\in B} |\mathfrak{E}(\theta) - \mathcal{E}(H_\theta)| \geq \varepsilon\big) \leq 2\mathcal{C}_{(B,\delta), \frac{\varepsilon}{8L\mathcal{R}}} \exp\left(\frac{-\varepsilon^2 M}{2\mathcal{R}^4}\right) \tag{211}$$

(cf. Definition 3.11). Moreover, note that Proposition 3.12 (with $X \leftarrow \mathbb{R}^{\mathfrak{d}}$, $\|\cdot\| \leftarrow (\mathbb{R}^{\mathfrak{d}} \ni x \mapsto |||x||| \in [0, \infty))$, $R \leftarrow R$, $r \leftarrow \frac{\varepsilon}{8L\mathcal{R}}$, $B \leftarrow B$, $\delta \leftarrow \delta$ in the notation of Proposition 3.12) demonstrates that

$$\mathcal{C}_{(B,\delta), \frac{\varepsilon}{8L\mathcal{R}}} \leq \max\left\{1, \left[\frac{32LR\mathcal{R}}{\varepsilon}\right]^{\mathfrak{d}}\right\}. \tag{212}$$

This and (211) prove (209). The proof of Lemma 3.20 is thus completed. □

Lemma 3.21. *Let $\mathfrak{d}, M, L \in \mathbb{N}$, $u \in \mathbb{R}$, $v \in (u, \infty)$, $R \in [1, \infty)$, $\varepsilon, b \in (0, \infty)$, $l = (l_0, l_1, \dots, l_L) \in \mathbb{N}^{L+1}$ satisfy $l_L = 1$ and $\sum_{k=1}^{L} l_k(l_{k-1} + 1) \le \mathfrak{d}$, let $D \subseteq [-b, b]^{l_0}$ be a compact set, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X_m\colon \Omega \to D$, $m \in \{1, 2, \dots, M\}$, and $Y_m\colon \Omega \to [u, v]$, $m \in \{1, 2, \dots, M\}$, be functions, assume that (X_m, Y_m), $m \in \{1, 2, \dots, M\}$, are i.i.d. random variables, let $\mathcal{E}\colon C(D, \mathbb{R}) \to [0, \infty)$ satisfy for all $f \in C(D, \mathbb{R})$ that $\mathcal{E}(f) = \mathbb{E}[|f(X_1) - Y_1|^2]$, and let $\mathfrak{E}\colon [-R, R]^{\mathfrak{d}} \times \Omega \to [0, \infty)$ satisfy for all $\theta \in [-R, R]^{\mathfrak{d}}$, $\omega \in \Omega$ that*

$$\mathfrak{E}(\theta, \omega) = \frac{1}{M}\left[\sum_{m=1}^{M} |\mathscr{N}_{u,v}^{\theta,l}(X_m(\omega)) - Y_m(\omega)|^2\right] \tag{213}$$

(cf. Definitions 2.20 and 2.8). Then $\Omega \ni \omega \mapsto \sup_{\theta \in [-R,R]^{\mathfrak{d}}} \big|\mathfrak{E}(\theta, \omega) - \mathcal{E}\big(\mathscr{N}_{u,v}^{\theta,l}|_D\big)\big| \in [0, \infty]$ is an $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable function and

$$\begin{aligned} &\mathbb{P}\big(\sup_{\theta \in [-R,R]^{\mathfrak{d}}} \big|\mathfrak{E}(\theta) - \mathcal{E}\big(\mathscr{N}_{u,v}^{\theta,l}|_D\big)\big| \ge \varepsilon\big) \\ &\le 2\max\left\{1, \left[\frac{32L\max\{1, b\}(|||l||| + 1)^L R^L (v - u)}{\varepsilon}\right]^{\mathfrak{d}}\right\} \exp\left(\frac{-\varepsilon^2 M}{2(v - u)^4}\right). \end{aligned} \tag{214}$$

Proof of Lemma 3.21. Throughout this proof let $\mathfrak{L} \in (0, \infty)$ satisfy

$$\mathfrak{L} = L\max\{1, b\}\,(|||l||| + 1)^L R^{L-1}. \tag{215}$$

Observe that Corollary 2.37 (with $a \leftarrow -b$, $b \leftarrow b$, $u \leftarrow u$, $v \leftarrow v$, $d \leftarrow \mathfrak{d}$, $L \leftarrow L$, $l \leftarrow l$ in the notation of Corollary 2.37) and the hypothesis that $D \subseteq [-b, b]^{l_0}$ show that for all $\theta, \vartheta \in [-R, R]^{\mathfrak{d}}$ it holds that

$$\begin{aligned} \sup_{x \in D} |\mathscr{N}_{u,v}^{\theta,l}(x) - \mathscr{N}_{u,v}^{\vartheta,l}(x)| &\le \sup_{x \in [-b,b]^{l_0}} |\mathscr{N}_{u,v}^{\theta,l}(x) - \mathscr{N}_{u,v}^{\vartheta,l}(x)| \\ &\le L\max\{1, b\}\,(|||l||| + 1)^L\,(\max\{1, |||\theta|||, |||\vartheta|||\})^{L-1} |||\theta - \vartheta||| \\ &\le L\max\{1, b\}\,(|||l||| + 1)^L R^{L-1} |||\theta - \vartheta||| = \mathfrak{L}|||\theta - \vartheta|||. \end{aligned} \tag{216}$$

Furthermore, observe that the fact that for all $\theta \in \mathbb{R}^{\mathfrak{d}}$, $x \in \mathbb{R}^{l_0}$ it holds that $\mathscr{N}_{u,v}^{\theta,l}(x) \in [u, v]$ and the hypothesis that for all $m \in \{1, 2, \dots, M\}$, $\omega \in \Omega$ it holds that $Y_m(\omega) \in [u, v]$ demonstrate that for all $\theta \in [-R, R]^{\mathfrak{d}}$, $m \in \{1, 2, \dots, M\}$ it holds that

$$|\mathscr{N}_{u,v}^{\theta,l}(X_m) - Y_m| \le v - u. \tag{217}$$

Combining this and (216) with Lemma 3.20 (with $d \leftarrow l_0$, $\mathfrak{d} \leftarrow \mathfrak{d}$, $M \leftarrow M$, $R \leftarrow R$, $L \leftarrow \mathfrak{L}$, $\mathcal{R} \leftarrow v - u$, $\varepsilon \leftarrow \varepsilon$, $D \leftarrow D$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(X_m)_{m \in \{1,2,\dots,M\}} \leftarrow (X_m)_{m \in \{1,2,\dots,M\}}$, $(Y_m)_{m \in \{1,2,\dots,M\}} \leftarrow ((\Omega \ni \omega \mapsto Y_m(\omega) \in \mathbb{R}))_{m \in \{1,2,\dots,M\}}$, $H \leftarrow ([-R, R]^{\mathfrak{d}} \ni \theta \mapsto \mathscr{N}_{u,v}^{\theta,l}|_D \in C(D, \mathbb{R}))$, $\mathcal{E} \leftarrow \mathcal{E}$, $\mathfrak{E} \leftarrow \mathfrak{E}$ in the notation of Lemma 3.20) establishes that $\Omega \ni \omega \mapsto \sup_{\theta \in [-R,R]^{\mathfrak{d}}} \big|\mathfrak{E}(\theta, \omega) - \mathcal{E}\big(\mathscr{N}_{u,v}^{\theta,l}|_D\big)\big| \in [0, \infty]$ is an $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable function and

$$\mathbb{P}\big(\sup_{\theta \in [-R,R]^{\mathfrak{d}}} \big|\mathfrak{E}(\theta) - \mathcal{E}\big(\mathscr{N}_{u,v}^{\theta,l}|_D\big)\big| \ge \varepsilon\big) \le 2\max\left\{1, \left[\frac{32\mathfrak{L}R(v - u)}{\varepsilon}\right]^{\mathfrak{d}}\right\} \exp\left(\frac{-\varepsilon^2 M}{2(v - u)^4}\right). \tag{218}$$

The proof of Lemma 3.21 is thus completed. ∎

3.3 Analysis of the optimization error

3.3.1 Convergence rates for the minimum Monte Carlo method

Lemma 3.22. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathfrak{d}, N \in \mathbb{N}$, let $\|\cdot\| : \mathbb{R}^{\mathfrak{d}} \to [0, \infty)$ be a norm, let $\mathfrak{H} \subseteq \mathbb{R}^{\mathfrak{d}}$ be a set, let $\vartheta \in \mathfrak{H}$, $L, \varepsilon \in (0, \infty)$, let $\mathfrak{E}\colon \mathfrak{H} \times \Omega \to \mathbb{R}$ be a $(\mathcal{B}(\mathfrak{H}) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable function, assume for all $x, y \in \mathfrak{H}$, $\omega \in \Omega$ that $|\mathfrak{E}(x, \omega) - \mathfrak{E}(y, \omega)| \le L\|x - y\|$, and let $\Theta_n\colon \Omega \to \mathfrak{H}$, $n \in \{1, 2, \dots, N\}$, be i.i.d. random variables. Then*

$$\mathbb{P}\big(\big[\min_{n \in \{1,2,\dots,N\}} \mathfrak{E}(\Theta_n)\big] - \mathfrak{E}(\vartheta) > \varepsilon\big) \le \big[\mathbb{P}\big(\|\Theta_1 - \vartheta\| > \tfrac{\varepsilon}{L}\big)\big]^N \le \exp\big(-N\,\mathbb{P}\big(\|\Theta_1 - \vartheta\| \le \tfrac{\varepsilon}{L}\big)\big). \tag{219}$$

Proof of Lemma 3.22. Note that the hypothesis that for all $x, y \in \mathfrak{H}$, $\omega \in \Omega$ it holds that $|\mathfrak{E}(x,\omega) - \mathfrak{E}(y,\omega)| \leq L\|x-y\|$ implies that

$$\begin{aligned}&\big[\min\nolimits_{n\in\{1,2,\ldots,N\}} \mathfrak{E}(\Theta_n)\big] - \mathfrak{E}(\vartheta) = \min\nolimits_{n\in\{1,2,\ldots,N\}} [\mathfrak{E}(\Theta_n) - \mathfrak{E}(\vartheta)] \\ &\leq \min\nolimits_{n\in\{1,2,\ldots,N\}} |\mathfrak{E}(\Theta_n) - \mathfrak{E}(\vartheta)| \leq \min\nolimits_{n\in\{1,2,\ldots,N\}} \big[L\|\Theta_n - \vartheta\|\big] \\ &= L\big[\min\nolimits_{n\in\{1,2,\ldots,N\}} \|\Theta_n - \vartheta\|\big].\end{aligned} \tag{220}$$

The hypothesis that Θ_n, $n \in \{1,2,\ldots,N\}$, are i.i.d. random variables and the fact that $\forall\, x \in \mathbb{R}\colon 1 - x \leq e^{-x}$ hence show that

$$\begin{aligned}&\mathbb{P}\Big(\big[\min\nolimits_{n\in\{1,2,\ldots,N\}} \mathfrak{E}(\Theta_n)\big] - \mathfrak{E}(\vartheta) > \varepsilon\Big) \leq \mathbb{P}\Big(L\big[\min\nolimits_{n\in\{1,2,\ldots,N\}} \|\Theta_n - \vartheta\|\big] > \varepsilon\Big) \\ &= \mathbb{P}\big(\min\nolimits_{n\in\{1,2,\ldots,N\}} \|\Theta_n - \vartheta\| > \tfrac{\varepsilon}{L}\big) = \big[\mathbb{P}\big(\|\Theta_1 - \vartheta\| > \tfrac{\varepsilon}{L}\big)\big]^N \\ &= \big[1 - \mathbb{P}\big(\|\Theta_1 - \vartheta\| \leq \tfrac{\varepsilon}{L}\big)\big]^N \leq \exp\!\big(-N\,\mathbb{P}\big(\|\Theta_1 - \vartheta\| \leq \tfrac{\varepsilon}{L}\big)\big).\end{aligned} \tag{221}$$

The proof of Lemma 3.22 is thus completed. ☐

3.3.2 Continuous uniformly distributed samples

Lemma 3.23. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathfrak{d}, N \in \mathbb{N}$, $a \in \mathbb{R}$, $b \in (a,\infty)$, $\vartheta \in [a,b]^{\mathfrak{d}}$, $L, \varepsilon \in (0,\infty)$, let $\mathfrak{E}\colon [a,b]^{\mathfrak{d}} \times \Omega \to \mathbb{R}$ be a $(\mathcal{B}([a,b]^{\mathfrak{d}}) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable function, assume for all $x, y \in [a,b]^{\mathfrak{d}}$, $\omega \in \Omega$ that $|\mathfrak{E}(x,\omega) - \mathfrak{E}(y,\omega)| \leq L|||x-y|||$, let $\Theta_n\colon \Omega \to [a,b]^{\mathfrak{d}}$, $n \in \{1,2,\ldots,N\}$, be i.i.d. random variables, and assume that Θ_1 is continuous uniformly distributed on $[a,b]^{\mathfrak{d}}$ (cf. Definition 2.20). Then*

$$\mathbb{P}\big(\big[\min\nolimits_{n\in\{1,2,\ldots,N\}} \mathfrak{E}(\Theta_n)\big] - \mathfrak{E}(\vartheta) > \varepsilon\big) \leq \exp\!\left(-N \min\!\left\{1, \frac{\varepsilon^{\mathfrak{d}}}{L^{\mathfrak{d}}(b-a)^{\mathfrak{d}}}\right\}\right). \tag{222}$$

Proof of Lemma 3.23. Note that the hypothesis that Θ_1 is continuous uniformly distributed on $[a,b]^{\mathfrak{d}}$ ensures that

$$\begin{aligned}&\mathbb{P}\big(|||\Theta_1 - \vartheta||| \leq \tfrac{\varepsilon}{L}\big) \geq \mathbb{P}\big(|||\Theta_1 - (a,a,\ldots,a)||| \leq \tfrac{\varepsilon}{L}\big) = \mathbb{P}\big(|||\Theta_1 - (a,a,\ldots,a)||| \leq \min\{\tfrac{\varepsilon}{L}, b-a\}\big) \\ &= \left[\frac{\min\{\frac{\varepsilon}{L}, b-a\}}{(b-a)}\right]^{\mathfrak{d}} = \min\!\left\{1, \left[\frac{\varepsilon}{L\,(b-a)}\right]^{\mathfrak{d}}\right\}.\end{aligned} \tag{223}$$

Combining this with Lemma 3.22 proves (222). The proof of Lemma 3.23 is thus completed. ☐

4 Overall error analysis

In this section we combine the separate error analyses of the approximation error, the generalization error, and the optimization error in Section 3 to obtain an overall analysis (cf. Theorem 4.5 below). We note that, e.g., [6, Lemma 2.4] ensures that the integral appearing on the left-hand side of (238) in Theorem 4.5 and subsequent results (cf. (251) in Corollary 4.6, (259) in Corollary 4.7, (269) in Corollary 4.8, and (274) in Corollary 4.10) is indeed measurable. In Lemma 4.1 below we present the well-known bias-variance decomposition result. To formulate this bias-variance decomposition lemma we observe that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every measurable space $(S, \mathcal{S})$, every random variable $X\colon \Omega \to S$, and every $A \in \mathcal{S}$ it holds that $\mathbb{P}_X(A) = \mathbb{P}(X \in A)$. Moreover, note that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every measurable space $(S, \mathcal{S})$, every random variable $X\colon \Omega \to S$, and every $\mathcal{S}/\mathcal{B}(\mathbb{R})$-measurable function $f\colon S \to \mathbb{R}$ it holds that $\int_S |f|^2 \, d\mathbb{P}_X = \int_S |f(x)|^2 \, \mathbb{P}_X(dx) = \int_\Omega |f(X(\omega))|^2 \, \mathbb{P}(d\omega) = \int_\Omega |f(X)|^2 \, d\mathbb{P} = \mathbb{E}\big[|f(X)|^2\big]$. A result related to Lemmas 4.1 and 4.2 can, e.g., be found in Berner et al. [10, Lemma 2.8].

4.1 Bias-variance decomposition

Lemma 4.1 (Bias-variance decomposition). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(S, \mathcal{S})$ be a measurable space, let $X\colon \Omega \to S$ and $Y\colon \Omega \to \mathbb{R}$ be random variables with $\mathbb{E}[|Y|^2] < \infty$, and let $\mathcal{E}\colon \mathcal{L}^2(\mathbb{P}_X;\mathbb{R}) \to [0,\infty)$ satisfy for all $f \in \mathcal{L}^2(\mathbb{P}_X;\mathbb{R})$ that $\mathcal{E}(f) = \mathbb{E}[|f(X) - Y|^2]$. Then*

(i) it holds for all $f \in \mathcal{L}^2(\mathbb{P}_X;\mathbb{R})$ that

$$\mathcal{E}(f) = \mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] + \mathbb{E}\big[|Y - \mathbb{E}[Y|X]|^2\big], \tag{224}$$

(ii) it holds for all $f, g \in \mathcal{L}^2(\mathbb{P}_X;\mathbb{R})$ that

$$\mathcal{E}(f) - \mathcal{E}(g) = \mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] - \mathbb{E}\big[|g(X) - \mathbb{E}[Y|X]|^2\big], \tag{225}$$

and

(iii) it holds for all $f, g \in \mathcal{L}^2(\mathbb{P}_X;\mathbb{R})$ that

$$\mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] = \mathbb{E}\big[|g(X) - \mathbb{E}[Y|X]|^2\big] + \big(\mathcal{E}(f) - \mathcal{E}(g)\big). \tag{226}$$

Proof of Lemma 4.1. First, observe that the hypothesis that for all $f \in \mathcal{L}^2(\mathbb{P}_X;\mathbb{R})$ it holds that $\mathcal{E}(f) = \mathbb{E}[|f(X) - Y|^2]$ shows that for all $f \in \mathcal{L}^2(\mathbb{P}_X;\mathbb{R})$ it holds that

$$\begin{aligned}
\mathcal{E}(f) &= \mathbb{E}\big[|f(X) - Y|^2\big] = \mathbb{E}\big[|(f(X) - \mathbb{E}[Y|X]) + (\mathbb{E}[Y|X] - Y)|^2\big] \\
&= \mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] + 2\,\mathbb{E}\big[\big(f(X) - \mathbb{E}[Y|X]\big)\big(\mathbb{E}[Y|X] - Y\big)\big] + \mathbb{E}\big[|\mathbb{E}[Y|X] - Y|^2\big] \\
&= \mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] + 2\,\mathbb{E}\Big[\mathbb{E}\big[\big(f(X) - \mathbb{E}[Y|X]\big)\big(\mathbb{E}[Y|X] - Y\big)\big|X\big]\Big] + \mathbb{E}\big[|\mathbb{E}[Y|X] - Y|^2\big] \\
&= \mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] + 2\,\mathbb{E}\Big[\big(f(X) - \mathbb{E}[Y|X]\big)\mathbb{E}\big[\big(\mathbb{E}[Y|X] - Y\big)\big|X\big]\Big] + \mathbb{E}\big[|\mathbb{E}[Y|X] - Y|^2\big] \\
&= \mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] + 2\,\mathbb{E}\big[\big(f(X) - \mathbb{E}[Y|X]\big)\big(\mathbb{E}[Y|X] - \mathbb{E}[Y|X]\big)\big] + \mathbb{E}\big[|\mathbb{E}[Y|X] - Y|^2\big] \\
&= \mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] + \mathbb{E}\big[|\mathbb{E}[Y|X] - Y|^2\big].
\end{aligned} \tag{227}$$

This implies that for all $f, g \in \mathcal{L}^2(\mathbb{P}_X;\mathbb{R})$ it holds that

$$\mathcal{E}(f) - \mathcal{E}(g) = \mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] - \mathbb{E}\big[|g(X) - \mathbb{E}[Y|X]|^2\big]. \tag{228}$$

Hence, we obtain that for all $f, g \in \mathcal{L}^2(\mathbb{P}_X;\mathbb{R})$ it holds that

$$\mathbb{E}\big[|f(X) - \mathbb{E}[Y|X]|^2\big] = \mathbb{E}\big[|g(X) - \mathbb{E}[Y|X]|^2\big] + \mathcal{E}(f) - \mathcal{E}(g). \tag{229}$$

Combining this with (227) and (228) establishes items (i), (ii), and (iii). The proof of Lemma 4.1 is thus completed. ☐

4.2 Overall error decomposition

Lemma 4.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $d, M \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be a compact set, let $X_m\colon \Omega \to D$, $m \in \{1, 2, \ldots, M\}$, and $Y_m\colon \Omega \to \mathbb{R}$, $m \in \{1, 2, \ldots, M\}$, be functions, assume that (X_m, Y_m), $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables, assume $\mathbb{E}[|Y_1|^2] < \infty$, let $\mathcal{E}\colon C(D,\mathbb{R}) \to [0,\infty)$ satisfy for all $f \in C(D,\mathbb{R})$ that $\mathcal{E}(f) = \mathbb{E}[|f(X_1) - Y_1|^2]$, and let $\mathfrak{E}\colon C(D,\mathbb{R}) \times \Omega \to [0,\infty)$ satisfy for all $f \in C(D,\mathbb{R})$, $\omega \in \Omega$ that*

$$\mathfrak{E}(f, \omega) = \frac{1}{M}\left[\sum_{m=1}^{M} |f(X_m(\omega)) - Y_m(\omega)|^2\right]. \tag{230}$$

Then it holds for all $f, \phi \in C(D,\mathbb{R})$ that

$$\begin{aligned}
\mathbb{E}\big[|f(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] &= \mathbb{E}\big[|\phi(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \mathcal{E}(f) - \mathcal{E}(\phi) \\
&\leq \mathbb{E}\big[|\phi(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \big[\mathfrak{E}(f) - \mathfrak{E}(\phi)\big] + 2\left[\max_{v \in \{f,\phi\}} |\mathfrak{E}(v) - \mathcal{E}(v)|\right].
\end{aligned} \tag{231}$$

Proof of Lemma 4.2. Note that Lemma 4.1 ensures that for all $f, \phi \in C(D, \mathbb{R})$ it holds that

$$\begin{aligned}
&\mathbb{E}\big[|f(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] \\
&= \mathbb{E}\big[|\phi(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \mathcal{E}(f) - \mathcal{E}(\phi) \\
&= \mathbb{E}\big[|\phi(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \mathcal{E}(f) - \mathfrak{E}(f) + \mathfrak{E}(f) - \mathfrak{E}(\phi) + \mathfrak{E}(\phi) - \mathcal{E}(\phi) \\
&= \mathbb{E}\big[|\phi(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \big[\big(\mathcal{E}(f) - \mathfrak{E}(f)\big) + \big(\mathfrak{E}(\phi) - \mathcal{E}(\phi)\big)\big] + \big[\mathfrak{E}(f) - \mathfrak{E}(\phi)\big] \\
&\leq \mathbb{E}\big[|\phi(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \left[\sum_{v \in \{f, \phi\}} |\mathfrak{E}(v) - \mathcal{E}(v)|\right] + \big[\mathfrak{E}(f) - \mathfrak{E}(\phi)\big] \\
&\leq \mathbb{E}\big[|\phi(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + 2\left[\max_{v \in \{f, \phi\}} |\mathfrak{E}(v) - \mathcal{E}(v)|\right] + \big[\mathfrak{E}(f) - \mathfrak{E}(\phi)\big].
\end{aligned} \tag{232}$$

The proof of Lemma 4.2 is thus completed. □

Lemma 4.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $d, \mathfrak{d}, M \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be a compact set, let $B \subseteq \mathbb{R}^{\mathfrak{d}}$ be a set, let $H = (H_\theta)_{\theta \in B} \colon B \to C(D, \mathbb{R})$ be a function, let $X_m \colon \Omega \to D$, $m \in \{1, 2, \ldots, M\}$, and $Y_m \colon \Omega \to \mathbb{R}$, $m \in \{1, 2, \ldots, M\}$, be functions, assume that (X_m, Y_m), $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables, assume $\mathbb{E}[|Y_1|^2] < \infty$, let $\varphi \colon D \to \mathbb{R}$ be a $\mathcal{B}(D)/\mathcal{B}(\mathbb{R})$-measurable function, assume that it holds $\mathbb{P}$-a.s. that $\varphi(X_1) = \mathbb{E}[Y_1|X_1]$, let $\mathcal{E} \colon C(D, \mathbb{R}) \to [0, \infty)$ satisfy for all $f \in C(D, \mathbb{R})$ that $\mathcal{E}(f) = \mathbb{E}[|f(X_1) - Y_1|^2]$, and let $\mathfrak{E} \colon B \times \Omega \to [0, \infty)$ satisfy for all $\theta \in B$, $\omega \in \Omega$ that*

$$\mathfrak{E}(\theta, \omega) = \frac{1}{M}\left[\sum_{m=1}^{M} |H_\theta(X_m(\omega)) - Y_m(\omega)|^2\right]. \tag{233}$$

Then it holds for all $\theta, \vartheta \in B$ that

$$\begin{aligned}
\int_D |H_\theta(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) &= \int_D |H_\vartheta(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) + \mathcal{E}(H_\theta) - \mathcal{E}(H_\vartheta) \\
&\leq \int_D |H_\vartheta(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) + \big[\mathfrak{E}(\theta) - \mathfrak{E}(\vartheta)\big] + 2\left[\sup_{\eta \in B} |\mathfrak{E}(\eta) - \mathcal{E}(H_\eta)|\right].
\end{aligned} \tag{234}$$

Proof of Lemma 4.3. First, observe that Lemma 4.2 (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $d \leftarrow d$, $M \leftarrow M$, $D \leftarrow D$, $(X_m)_{m \in \{1,2,\ldots,M\}} \leftarrow (X_m)_{m \in \{1,2,\ldots,M\}}$, $(Y_m)_{m \in \{1,2,\ldots,M\}} \leftarrow (Y_m)_{m \in \{1,2,\ldots,M\}}$, $\mathcal{E} \leftarrow \mathcal{E}$, $\mathfrak{E} \leftarrow \big(C(D, \mathbb{R}) \times \Omega \ni (f, \omega) \mapsto \frac{1}{M}\big[\sum_{m=1}^{M} |f(X_m(\omega)) - Y_m(\omega)|^2\big] \in [0, \infty)\big)$ in the notation of Lemma 4.2) shows that for all $\theta, \vartheta \in B$ it holds that

$$\begin{aligned}
\mathbb{E}\big[|H_\theta(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] &= \mathbb{E}\big[|H_\vartheta(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \mathcal{E}(H_\theta) - \mathcal{E}(H_\vartheta) \\
&\leq \mathbb{E}\big[|H_\vartheta(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \big[\mathfrak{E}(\theta) - \mathfrak{E}(\vartheta)\big] + 2\left[\max_{\eta \in \{\theta, \vartheta\}} |\mathfrak{E}(\eta) - \mathcal{E}(H_\eta)|\right] \\
&\leq \mathbb{E}\big[|H_\vartheta(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] + \big[\mathfrak{E}(\theta) - \mathfrak{E}(\vartheta)\big] + 2\left[\sup_{\eta \in B} |\mathfrak{E}(\eta) - \mathcal{E}(H_\eta)|\right].
\end{aligned} \tag{235}$$

In addition, note that the hypothesis that it holds $\mathbb{P}$-a.s. that $\varphi(X_1) = \mathbb{E}[Y_1|X_1]$ ensures that for all $\eta \in B$ it holds that

$$\mathbb{E}\big[|H_\eta(X_1) - \mathbb{E}[Y_1|X_1]|^2\big] = \mathbb{E}\big[|H_\eta(X_1) - \varphi(X_1)|^2\big] = \int_D |H_\eta(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx). \tag{236}$$

Combining this with (235) establishes (234). The proof of Lemma 4.3 is thus completed. □

4.3 Analysis of the convergence speed

4.3.1 Convergence rates for convergence in probability

Lemma 4.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $u \in \mathbb{R}$, $v \in (u, \infty)$, $\mathfrak{d}, L \in \mathbb{N}$, let $l = (l_0, l_1, \ldots, l_L) \in \mathbb{N}^{L+1}$ satisfy $l_L = 1$ and $\sum_{i=1}^{L} l_i(l_{i-1}+1) \le \mathfrak{d}$, let $B \subseteq \mathbb{R}^{\mathfrak{d}}$ be a non-empty compact set, and let $X : \Omega \to \mathbb{R}^{l_0}$ and $Y : \Omega \to [u, v]$ be random variables. Then*

(i) it holds for all $\theta \in B$, $\omega \in \Omega$ that $|\mathscr{N}_{u,v}^{\theta,l}(X(\omega)) - Y(\omega)|^2 \in [0, (v-u)^2]$,

(ii) it holds that $B \ni \theta \mapsto \mathbb{E}\big[|\mathscr{N}_{u,v}^{\theta,l}(X) - Y|^2\big] \in [0, \infty)$ is continuous, and

(iii) there exists $\vartheta \in B$ such that $\mathbb{E}\big[|\mathscr{N}_{u,v}^{\vartheta,l}(X) - Y|^2\big] = \inf_{\theta \in B} \mathbb{E}\big[|\mathscr{N}_{u,v}^{\theta,l}(X) - Y|^2\big]$

(cf. Definition 2.8).

Proof of Lemma 4.4. First, note that the fact that for all $\theta \in \mathbb{R}^{\mathfrak{d}}$, $x \in \mathbb{R}^{l_0}$ it holds that $\mathscr{N}_{u,v}^{\theta,l}(x) \in [u, v]$ and the hypothesis that for all $\omega \in \Omega$ it holds that $Y(\omega) \in [u, v]$ demonstrate item (i). Next observe that Corollary 2.37 ensures that for all $\omega \in \Omega$ it holds that $B \ni \theta \mapsto |\mathscr{N}_{u,v}^{\theta,l}(X(\omega)) - Y(\omega)|^2 \in [0, \infty)$ is a continuous function. Combining this and item (i) with Lebesgue's dominated convergence theorem establishes item (ii). Furthermore, note that item (ii) and the assumption that $B \subseteq \mathbb{R}^{\mathfrak{d}}$ is a non-empty compact set prove item (iii). The proof of Lemma 4.4 is thus completed. ☐

Theorem 4.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $d, \mathfrak{d}, K, M \in \mathbb{N}$, $\varepsilon \in (0, \infty)$, $L, u \in \mathbb{R}$, $v \in (u, \infty)$, let $D \subseteq \mathbb{R}^d$ be a compact set, assume $|D| \ge 2$, let $X_m : \Omega \to D$, $m \in \{1, 2, \ldots, M\}$, and $Y_m : \Omega \to [u, v]$, $m \in \{1, 2, \ldots, M\}$, be functions, assume that (X_m, Y_m), $m \in \{1, 2, \ldots, M\}$, are i.i.d. random variables, let $\delta : D \times D \to [0, \infty)$ satisfy for all $x = (x_1, x_2, \ldots, x_d), y = (y_1, y_2, \ldots, y_d) \in D$ that $\delta(x, y) = \sum_{i=1}^{d} |x_i - y_i|$, let $\varphi : D \to [u, v]$ satisfy $\mathbb{P}$-a.s. that $\varphi(X_1) = \mathbb{E}[Y_1|X_1]$, assume for all $x, y \in D$ that $|\varphi(x) - \varphi(y)| \le L\delta(x, y)$, let $N \in \mathbb{N} \cap [\max\{2, \mathcal{C}_{(D,\delta),\frac{\varepsilon}{4L}}\}, \infty)$, let $l \in \mathbb{N} \cap (N, \infty)$, let $\mathfrak{l} = (\mathfrak{l}_0, \mathfrak{l}_1, \ldots, \mathfrak{l}_l) \in \mathbb{N}^{l+1}$ satisfy for all $i \in \mathbb{N} \cap [2, N]$, $j \in \mathbb{N} \cap [N, l)$ that $\mathfrak{l}_0 = d$, $\mathfrak{l}_1 \ge 2dN$, $\mathfrak{l}_i \ge 2N - 2i + 3$, $\mathfrak{l}_j \ge 2$, $\mathfrak{l}_l = 1$, and $\sum_{k=1}^{l} \mathfrak{l}_k(\mathfrak{l}_{k-1} + 1) \le \mathfrak{d}$, let $R \in [\max\{1, L, \sup_{z \in D} |||z|||, 2[\sup_{z \in D} |\varphi(z)|]\}, \infty)$, let $B \subseteq \mathbb{R}^{\mathfrak{d}}$ satisfy $B = [-R, R]^{\mathfrak{d}}$, let $\mathfrak{E} : B \times \Omega \to [0, \infty)$ satisfy for all $\theta \in B$, $\omega \in \Omega$ that*

$$\mathfrak{E}(\theta, \omega) = \frac{1}{M}\left[\sum_{m=1}^{M} |\mathscr{N}_{u,v}^{\theta,\mathfrak{l}}(X_m(\omega)) - Y_m(\omega)|^2\right], \tag{237}$$

let $\Theta_k : \Omega \to B$, $k \in \{1, 2, \ldots, K\}$, be i.i.d. random variables, assume that Θ_1 is continuous uniformly distributed on B, and let $\Xi : \Omega \to B$ satisfy $\Xi = \Theta_{\min\{k \in \{1,2,\ldots,K\} : \mathfrak{E}(\Theta_k) = \min_{l \in \{1,2,\ldots,K\}} \mathfrak{E}(\Theta_l)\}}$ (cf. Definitions 3.11, 2.20, and 2.8). Then

$$\begin{aligned}\mathbb{P}\left(\int_D |\mathscr{N}_{u,v}^{\Xi,\mathfrak{l}}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) > \varepsilon^2\right) &\le \exp\left(-K \min\left\{1, \frac{\varepsilon^{2\mathfrak{d}}}{(16(v-u)l(|||\mathfrak{l}||| + 1)^l R^{l+1})^{\mathfrak{d}}}\right\}\right) \\ &\quad + 2\exp\left(\mathfrak{d} \ln\left(\max\left\{1, \frac{128l(|||\mathfrak{l}||| + 1)^l R^{l+1}(v-u)}{\varepsilon^2}\right\}\right) - \frac{\varepsilon^4 M}{32(v-u)^4}\right).\end{aligned} \tag{238}$$

Proof of Theorem 4.5. Throughout this proof let $\mathcal{M} \subseteq D$ satisfy $|\mathcal{M}| = \max\{2, \mathcal{C}_{(D,\delta),\frac{\varepsilon}{4L}}\}$ and

$$4L\left[\sup_{x \in D}\left(\inf_{y \in \mathcal{M}} \delta(x, y)\right)\right] \le \varepsilon, \tag{239}$$

let $b \in [0, \infty)$ satisfy $b = \sup_{z \in D} |||z|||$, let $\mathcal{E} : C(D, \mathbb{R}) \to [0, \infty)$ satisfy for all $f \in C(D, \mathbb{R})$ that $\mathcal{E}(f) = \mathbb{E}[|f(X_1) - Y_1|^2]$, and let $\vartheta \in B$ satisfy $\mathcal{E}(\mathscr{N}_{u,v}^{\vartheta,\mathfrak{l}}|_D) = \inf_{\theta \in B} \mathcal{E}(\mathscr{N}_{u,v}^{\theta,\mathfrak{l}}|_D)$ (cf. Lemma 4.4). Observe that the hypothesis that for all $x, y \in D$ it holds that $|\varphi(x) - \varphi(y)| \le L\delta(x, y)$ implies that φ is a $\mathcal{B}(D)/\mathcal{B}([u, v])$-measurable function. Lemma 4.3 (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $d \leftarrow d$, $\mathfrak{d} \leftarrow \mathfrak{d}$, $M \leftarrow M$, $D \leftarrow D$, $B \leftarrow B$,

$H \leftarrow (B \ni \theta \mapsto \mathscr{N}^{\theta,\mathfrak{l}}_{u,v}|_D \in C(D,\mathbb{R}))$, $(X_m)_{m\in\{1,2,\ldots,M\}} \leftarrow (X_m)_{m\in\{1,2,\ldots,M\}}$, $(Y_m)_{m\in\{1,2,\ldots,M\}} \leftarrow ((\Omega \ni \omega \mapsto Y_m(\omega) \in \mathbb{R}))_{m\in\{1,2,\ldots,M\}}$, $\varphi \leftarrow (D \ni x \mapsto \varphi(x) \in \mathbb{R})$, $\mathcal{E} \leftarrow \mathcal{E}$, $\mathfrak{E} \leftarrow \mathfrak{E}$ in the notation of Lemma 4.3) therefore ensures that for all $\omega \in \Omega$ it holds that

$$\begin{aligned}
&\int_D |\mathscr{N}^{\Xi(\omega),\mathfrak{l}}_{u,v}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) \\
&\leq \underbrace{\int_D |\mathscr{N}^{\vartheta,\mathfrak{l}}_{u,v}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx)}_{\text{Approximation error}} + \underbrace{\big[\mathfrak{E}(\Xi(\omega),\omega) - \mathfrak{E}(\vartheta,\omega)\big]}_{\text{Optimization error}} + \underbrace{2\bigg[\sup_{\theta\in B} |\mathfrak{E}(\theta,\omega) - \mathcal{E}(\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}|_D)|\bigg]}_{\text{Generalization error}}.
\end{aligned} \tag{240}$$

Next observe that the assumption that $N \geq \max\{2, \mathcal{C}_{(D,\delta),\frac{\varepsilon}{4L}}\} = |\mathcal{M}|$ shows that for all $i \in \mathbb{N} \cap [2, N]$ it holds that $l \geq |\mathcal{M}| + 1$, $\mathfrak{l}_1 \geq 2d|\mathcal{M}|$ and $\mathfrak{l}_i \geq 2|\mathcal{M}| - 2i + 3$. The hypothesis that for all $x, y \in D$ it holds that $|\varphi(x) - \varphi(y)| \leq L\delta(x, y)$, the hypothesis that $R \geq \max\{1, L, \sup_{z\in D} |||z|||, 2[\sup_{z\in D} |\varphi(z)|]\}$, Corollary 3.8 (with $d \leftarrow d$, $\mathfrak{d} \leftarrow \mathfrak{d}$, $\mathfrak{L} \leftarrow l$, $L \leftarrow L$, $u \leftarrow u$, $v \leftarrow v$, $D \leftarrow D$, $f \leftarrow \varphi$, $\mathcal{M} \leftarrow \mathcal{M}$, $l \leftarrow \mathfrak{l}$ in the notation of Corollary 3.8), and (239) hence ensure that there exists $\eta \in B$ which satisfies

$$\begin{aligned}
\sup_{x\in D} |\mathscr{N}^{\eta,\mathfrak{l}}_{u,v}(x) - \varphi(x)| &\leq 2L\left[\sup_{x=(x_1,x_2,\ldots,x_d)\in D} \left(\inf_{y=(y_1,y_2,\ldots,y_d)\in\mathcal{M}} \sum_{i=1}^d |x_i - y_i|\right)\right] \\
&= 2L\left[\sup_{x\in D}\left(\inf_{y\in\mathcal{M}} \delta(x,y)\right)\right] \leq \frac{\varepsilon}{2}.
\end{aligned} \tag{241}$$

Lemma 4.3 (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $d \leftarrow d$, $\mathfrak{d} \leftarrow \mathfrak{d}$, $M \leftarrow M$, $D \leftarrow D$, $B \leftarrow B$, $H \leftarrow (B \ni \theta \mapsto \mathscr{N}^{\theta,\mathfrak{l}}_{u,v}|_D \in C(D,\mathbb{R}))$, $(X_m)_{m\in\{1,2,\ldots,M\}} \leftarrow (X_m)_{m\in\{1,2,\ldots,M\}}$, $(Y_m)_{m\in\{1,2,\ldots,M\}} \leftarrow ((\Omega \ni \omega \mapsto Y_m(\omega) \in \mathbb{R}))_{m\in\{1,2,\ldots,M\}}$, $\varphi \leftarrow (D \ni x \mapsto \varphi(x) \in \mathbb{R})$, $\mathcal{E} \leftarrow \mathcal{E}$, $\mathfrak{E} \leftarrow \mathfrak{E}$ in the notation of Lemma 4.3) and the assumption that $\mathcal{E}(\mathscr{N}^{\vartheta,\mathfrak{l}}_{u,v}|_D) = \inf_{\theta\in B} \mathcal{E}(\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}|_D)$ therefore prove that

$$\begin{aligned}
&\int_D |\mathscr{N}^{\vartheta,\mathfrak{l}}_{u,v}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) = \int_D |\mathscr{N}^{\eta,\mathfrak{l}}_{u,v}(x) - \varphi|^2 \, \mathbb{P}_{X_1}(dx) + \underbrace{\mathcal{E}(\mathscr{N}^{\vartheta,\mathfrak{l}}_{u,v}|_D) - \mathcal{E}(\mathscr{N}^{\eta,\mathfrak{l}}_{u,v}|_D)}_{\leq 0} \\
&\leq \int_D |\mathscr{N}^{\eta,\mathfrak{l}}_{u,v}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) \leq \sup_{x\in D} |\mathscr{N}^{\eta,\mathfrak{l}}_{u,v}(x) - \varphi(x)|^2 \leq \frac{\varepsilon^2}{4}.
\end{aligned} \tag{242}$$

Combining this with (240) shows that for all $\omega \in \Omega$ it holds that

$$\int_D |\mathscr{N}^{\Xi(\omega),\mathfrak{l}}_{u,v}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) \leq \frac{\varepsilon^2}{4} + \big[\mathfrak{E}(\Xi(\omega),\omega) - \mathfrak{E}(\vartheta,\omega)\big] + 2\bigg[\sup_{\theta\in B} |\mathfrak{E}(\theta,\omega) - \mathcal{E}(\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}|_D)|\bigg]. \tag{243}$$

Hence, we obtain that

$$\begin{aligned}
&\mathbb{P}\bigg(\int_D |\mathscr{N}^{\Xi,\mathfrak{l}}_{u,v}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) > \varepsilon^2\bigg) \leq \mathbb{P}\bigg(\big[\mathfrak{E}(\Xi) - \mathfrak{E}(\vartheta)\big] + 2\bigg[\sup_{\theta\in B} |\mathfrak{E}(\theta) - \mathcal{E}(\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}|_D)|\bigg] > \frac{3\varepsilon^2}{4}\bigg) \\
&\leq \mathbb{P}\bigg(\mathfrak{E}(\Xi) - \mathfrak{E}(\vartheta) > \frac{\varepsilon^2}{4}\bigg) + \mathbb{P}\bigg(\sup_{\theta\in B} |\mathfrak{E}(\theta) - \mathcal{E}(\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}|_D)| > \frac{\varepsilon^2}{4}\bigg).
\end{aligned} \tag{244}$$

Next observe that Corollary 2.37 (with $a \leftarrow -b$, $b \leftarrow b$, $u \leftarrow u$, $v \leftarrow v$, $d \leftarrow \mathfrak{d}$, $L \leftarrow l$, $l \leftarrow \mathfrak{l}$ in the notation of Corollary 2.37) demonstrates that for all $\theta, \xi \in B$ it holds that

$$\begin{aligned}
\sup_{x\in D} |\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}(x) - \mathscr{N}^{\xi,\mathfrak{l}}_{u,v}(x)| &\leq \sup_{x\in[-b,b]^d} |\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}(x) - \mathscr{N}^{\xi,\mathfrak{l}}_{u,v}(x)| \\
&\leq l \max\{1, b\}(|||\mathfrak{l}||| + 1)^l (\max\{1, |||\theta|||, |||\xi|||\})^{l-1} |||\theta - \xi||| \\
&\leq lR(|||\mathfrak{l}||| + 1)^l R^{l-1} |||\theta - \xi||| = l(|||\mathfrak{l}||| + 1)^l R^l |||\theta - \xi|||.
\end{aligned} \tag{245}$$

Combining this with the fact that for all $\theta \in \mathbb{R}^{\mathfrak{d}}$, $x \in D$ it holds that $\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}(x) \in [u, v]$, the hypothesis that for all $m \in \{1, 2, \ldots, M\}$, $\omega \in \Omega$ it holds that $Y_m(\omega) \in [u, v]$, the fact that for all $x_1, x_2, y \in \mathbb{R}$ it holds that $(x_1 - y)^2 - (x_2 - y)^2 = (x_1 - x_2)((x_1 - y) + (x_2 - y))$, and (237) ensures that for all $\theta, \xi \in B$, $\omega \in \Omega$ it holds that

$$\begin{aligned}
&|\mathfrak{E}(\theta,\omega) - \mathfrak{E}(\xi,\omega)| \\
&= \left| \frac{1}{M}\left[\sum_{m=1}^{M} |\mathscr{N}_{u,v}^{\theta,\mathfrak{l}}(X_m(\omega)) - Y_m(\omega)|^2\right] - \frac{1}{M}\left[\sum_{m=1}^{M} |\mathscr{N}_{u,v}^{\xi,\mathfrak{l}}(X_m(\omega)) - Y_m(\omega)|^2\right]\right| \\
&= \frac{1}{M}\left|\sum_{m=1}^{M}\Big(\big(\mathscr{N}_{u,v}^{\theta,\mathfrak{l}}(X_m(\omega)) - \mathscr{N}_{u,v}^{\xi,\mathfrak{l}}(X_m(\omega))\big)\left[\big(\mathscr{N}_{u,v}^{\theta,\mathfrak{l}}(X_m(\omega)) - Y_m(\omega)\big) + \big(\mathscr{N}_{u,v}^{\xi,\mathfrak{l}}(X_m(\omega)) - Y_m(\omega)\big)\right]\Big)\right| \\
&\leq \frac{1}{M}\left[\sum_{m=1}^{M}\Big(\big|\mathscr{N}_{u,v}^{\theta,\mathfrak{l}}(X_m(\omega)) - \mathscr{N}_{u,v}^{\xi,\mathfrak{l}}(X_m(\omega))\big| \underbrace{\left[|\mathscr{N}_{u,v}^{\theta,\mathfrak{l}}(X_m(\omega)) - Y_m(\omega)| + |\mathscr{N}_{u,v}^{\xi,\mathfrak{l}}(X_m(\omega)) - Y_m(\omega)|\right]}_{\leq 2(v-u)}\Big)\right] \\
&\leq 2(v-u)l(|||\mathfrak{l}||| + 1)^l R^l |||\theta - \xi|||.
\end{aligned} \tag{246}$$

Lemma 3.23 (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $\mathfrak{d} \leftarrow \mathfrak{d}$, $N \leftarrow K$, $a \leftarrow -R$, $b \leftarrow R$, $\vartheta \leftarrow \vartheta$, $L \leftarrow 2(v-u)l(|||\mathfrak{l}||| + 1)^l R^l$, $\varepsilon \leftarrow \frac{\varepsilon^2}{4}$, $\mathfrak{E} \leftarrow \mathfrak{E}$, $(\Theta_n)_{n\in\{1,2,\ldots,N\}} \leftarrow (\Theta_k)_{k\in\{1,2,\ldots,K\}}$ in the notation of Lemma 3.23) therefore shows that

$$\begin{aligned}
\mathbb{P}\bigg(\mathfrak{E}(\Xi) - \mathfrak{E}(\vartheta) > \frac{\varepsilon^2}{4}\bigg) &= \mathbb{P}\bigg(\Big[\min_{k\in\{1,2,\ldots,K\}} \mathfrak{E}(\Theta_k)\Big] - \mathfrak{E}(\vartheta) > \frac{\varepsilon^2}{4}\bigg) \\
&\leq \exp\left(-K \min\left\{1, \frac{\big(\frac{\varepsilon^2}{4}\big)^{\mathfrak{d}}}{[2(v-u)l(|||\mathfrak{l}||| + 1)^l R^l]^{\mathfrak{d}}(2R)^{\mathfrak{d}}}\right\}\right) \\
&= \exp\left(-K \min\left\{1, \frac{\varepsilon^{2\mathfrak{d}}}{(16(v-u)l(|||\mathfrak{l}||| + 1)^l R^{l+1})^{\mathfrak{d}}}\right\}\right).
\end{aligned} \tag{247}$$

Moreover, note that Lemma 3.21 (with $\mathfrak{d} \leftarrow \mathfrak{d}$, $M \leftarrow M$, $L \leftarrow l$, $u \leftarrow u$, $v \leftarrow v$, $R \leftarrow R$, $\varepsilon \leftarrow \frac{\varepsilon^2}{4}$, $b \leftarrow b$, $l \leftarrow \mathfrak{l}$, $D \leftarrow D$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(X_m)_{m\in\{1,2,\ldots,M\}} \leftarrow (X_m)_{m\in\{1,2,\ldots,M\}}$, $(Y_m)_{m\in\{1,2,\ldots,M\}} \leftarrow (Y_m)_{m\in\{1,2,\ldots,M\}}$, $\mathcal{E} \leftarrow \mathcal{E}$, $\mathfrak{E} \leftarrow \mathfrak{E}$ in the notation of Lemma 3.21) establishes that

$$\begin{aligned}
&\mathbb{P}\bigg(\sup\nolimits_{\theta\in B}|\mathfrak{E}(\theta) - \mathcal{E}(\mathscr{N}_{u,v}^{\theta,\mathfrak{l}}|_D)| \geq \frac{\varepsilon^2}{4}\bigg) \\
&\leq 2\max\left\{1, \left[\frac{128l\max\{1,b\}(|||\mathfrak{l}||| + 1)^l R^l (v-u)}{\varepsilon^2}\right]^{\mathfrak{d}}\right\} \exp\left(\frac{-\varepsilon^4 M}{32(v-u)^4}\right) \\
&\leq 2\max\left\{1, \left[\frac{128l(|||\mathfrak{l}||| + 1)^l R^{l+1}(v-u)}{\varepsilon^2}\right]^{\mathfrak{d}}\right\} \exp\left(\frac{-\varepsilon^4 M}{32(v-u)^4}\right) \\
&= 2\exp\left(\mathfrak{d}\ln\left(\max\left\{1, \frac{128l(|||\mathfrak{l}||| + 1)^l R^{l+1}(v-u)}{\varepsilon^2}\right\}\right) - \frac{\varepsilon^4 M}{32(v-u)^4}\right).
\end{aligned} \tag{248}$$

Combining this and (247) with (244) proves that

$$\begin{aligned}
\mathbb{P}\bigg(\int_D |\mathscr{N}_{u,v}^{\Xi,\mathfrak{l}}(x) - \varphi(x)|^2\, \mathbb{P}_{X_1}(dx) > \varepsilon^2\bigg) &\leq \exp\left(-K\min\left\{1, \frac{\varepsilon^{2\mathfrak{d}}}{(16(v-u)l(|||\mathfrak{l}||| + 1)^l R^{l+1})^{\mathfrak{d}}}\right\}\right) \\
&\quad + 2\exp\left(\mathfrak{d}\ln\left(\max\left\{1, \frac{128l(|||\mathfrak{l}||| + 1)^l R^{l+1}(v-u)}{\varepsilon^2}\right\}\right) - \frac{\varepsilon^4 M}{32(v-u)^4}\right).
\end{aligned} \tag{249}$$

The proof of Theorem 4.5 is thus completed. ☐

Corollary 4.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $d, \mathfrak{d}, K, M, \tau \in \mathbb{N}$, $\varepsilon \in (0,\infty)$, $L, a, u \in \mathbb{R}$, $b \in (a,\infty)$, $v \in (u,\infty)$, $R \in [\max\{1, L, |a|, |b|, 2|u|, 2|v|\}, \infty)$, let $X_m \colon \Omega \to [a,b]^d$, $m \in \{1,2,\ldots,M\}$, be i.i.d. random variables, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$, let $\varphi \colon [a,b]^d \to [u,v]$*

satisfy for all $x, y \in [a,b]^d$ *that* $|\varphi(x) - \varphi(y)| \leq L\|x - y\|$, *assume* $\tau \geq 2d(2dL(b-a)\varepsilon^{-1} + 2)^d$ *and* $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$, *let* $\mathfrak{l} \in \mathbb{N}^\tau$ *satisfy* $\mathfrak{l} = (d, \tau, \tau, \ldots, \tau, 1)$, *let* $B \subseteq \mathbb{R}^{\mathfrak{d}}$ *satisfy* $B = [-R, R]^{\mathfrak{d}}$, *let* $\mathfrak{E} \colon B \times \Omega \to [0, \infty)$ *satisfy for all* $\theta \in B$, $\omega \in \Omega$ *that*

$$\mathfrak{E}(\theta, \omega) = \frac{1}{M}\left[\sum_{m=1}^{M} |\mathscr{N}^{\theta,\mathfrak{l}}_{u,v}(X_m(\omega)) - \varphi(X_m(\omega))|^2\right], \tag{250}$$

let $\Theta_k \colon \Omega \to B$, $k \in \{1, 2, \ldots, K\}$, *be i.i.d. random variables, assume that* Θ_1 *is continuous uniformly distributed on* B, *and let* $\Xi \colon \Omega \to B$ *satisfy* $\Xi = \Theta_{\min\{k \in \{1,2,\ldots,K\} \colon \mathfrak{E}(\Theta_k) = \min_{l \in \{1,2,\ldots,K\}} \mathfrak{E}(\Theta_l)\}}$ *(cf. Definition 2.8). Then*

$$\mathbb{P}\left(\left[\int_{[a,b]^d} |\mathscr{N}^{\Xi,\mathfrak{l}}_{u,v}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx)\right]^{1/2} > \varepsilon\right) \leq \exp\left(-K \min\left\{1, \frac{\varepsilon^{2\mathfrak{d}}}{(16(v-u)(\tau+1)^\tau R^\tau)^{\mathfrak{d}}}\right\}\right) + 2\exp\left(\mathfrak{d} \ln\left(\max\left\{1, \frac{128(\tau+1)^\tau R^\tau (v-u)}{\varepsilon^2}\right\}\right) - \frac{\varepsilon^4 M}{32(v-u)^4}\right). \tag{251}$$

Proof of Corollary 4.6. Throughout this proof let $N \in \mathbb{N}$ satisfy

$$N = \min\left\{k \in \mathbb{N} \colon k \geq \frac{2dL(b-a)}{\varepsilon}\right\}, \tag{252}$$

let $\mathcal{M} \subseteq [a,b]^d$ satisfy $\mathcal{M} = \{a, a + \frac{b-a}{N}, \ldots, a + \frac{(N-1)(b-a)}{N}, b\}^d$, let $\delta \colon [a,b]^d \times [a,b]^d \to [0,\infty)$ satisfy for all $x = (x_1, x_2, \ldots, x_d), y = (y_1, y_2, \ldots, y_d) \in [a,b]^d$ that $\delta(x,y) = \sum_{i=1}^d |x_i - y_i|$, and let $l_0, l_1, \ldots, l_{\tau-1} \in \mathbb{N}$ satisfy $\mathfrak{l} = (l_0, l_1, \ldots, l_{\tau-1})$. Observe that for all $x \in [a,b]$ there exists $y \in \{a, a + \frac{b-a}{N}, \ldots, a + \frac{(N-1)(b-a)}{N}, b\}$ such that $|x - y| \leq \frac{b-a}{2N}$. This demonstrates that

$$4L\left[\sup_{x=(x_1,x_2,\ldots,x_d)\in[a,b]^d}\left(\inf_{y=(y_1,y_2,\ldots,y_d)\in\mathcal{M}} \sum_{i=1}^d |x_i - y_i|\right)\right] \leq \frac{2Ld(b-a)}{N} \leq \varepsilon. \tag{253}$$

Hence, we obtain that

$$\mathcal{C}_{([a,b]^d,\delta),\frac{\varepsilon}{4L}} \leq |\mathcal{M}| = (N+1)^d. \tag{254}$$

Next note that (252) implies that $N < 2dL(b-a)\varepsilon^{-1} + 1$. The hypothesis that $\tau \geq 2d(2dL(b-a)\varepsilon^{-1} + 2)^d$ therefore ensures that

$$\tau > 2d(N+1)^d \geq (N+1)^d + 2. \tag{255}$$

Hence, we obtain that for all $i \in \{2, 3, \ldots, (N+1)^d\}$, $j \in \{(N+1)^d + 1, (N+1)^d + 2, \ldots, \tau - 2\}$ it holds that

$$l_0 = d, \quad l_1 = \tau \geq 2d(N+1)^d, \quad l_{\tau-1} = 1, \quad l_i = \tau \geq 2(N+1)^d - 2i + 3, \quad \text{and} \quad l_j = \tau \geq 2. \tag{256}$$

Furthermore, observe that the hypothesis that for all $x, y \in [a,b]^d$ it holds that $|\varphi(x) - \varphi(y)| \leq L\|x - y\|$ implies that for all $x, y \in [a,b]^d$ it holds that $|\varphi(x) - \varphi(y)| \leq L\delta(x,y)$. Combining this, (254), (255), (256), and the hypothesis that $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1 = \sum_{i=1}^{\tau-1} l_i(l_{i-1}+1)$ with Theorem 4.5 (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $d \leftarrow d$, $\mathfrak{d} \leftarrow \mathfrak{d}$, $K \leftarrow K$, $M \leftarrow M$, $\varepsilon \leftarrow \varepsilon$, $L \leftarrow L$, $u \leftarrow u$, $v \leftarrow v$, $D \leftarrow [a,b]^d$, $(X_m)_{m\in\{1,2,\ldots,M\}} \leftarrow (X_m)_{m\in\{1,2,\ldots,M\}}$, $(Y_m)_{m\in\{1,2,\ldots,M\}} \leftarrow (\varphi(X_m))_{m\in\{1,2,\ldots,M\}}$, $\delta \leftarrow \delta$, $\varphi \leftarrow \varphi$, $N \leftarrow (N+1)^d$, $l \leftarrow \tau - 1$, $\mathfrak{l} \leftarrow \mathfrak{l}$, $R \leftarrow R$, $B \leftarrow B$, $\mathfrak{E} \leftarrow \mathfrak{E}$, $(\Theta_k)_{k\in\{1,2,\ldots,K\}} \leftarrow (\Theta_k)_{k\in\{1,2,\ldots,K\}}$, $\Xi \leftarrow \Xi$ in the notation of

Theorem 4.5) establishes that

$$\begin{aligned}
&\mathbb{P}\Bigg(\bigg[\int_{[a,b]^d} |\mathscr{N}_{u,v}^{\Xi,\mathfrak{l}}(x) - \varphi(x)|^2 \,\mathbb{P}_{X_1}(dx)\bigg]^{1/2} > \varepsilon\Bigg) \\
&\leq \exp\!\left(-K \min\!\left\{1, \frac{\varepsilon^{2\mathfrak{d}}}{(16(v-u)(\tau-1)(\tau+1)^{\tau-1}R^\tau)^{\mathfrak{d}}}\right\}\right) \\
&\qquad + 2\exp\!\left(\mathfrak{d}\ln\!\left(\max\!\left\{1, \frac{128(\tau-1)(\tau+1)^{\tau-1}R^\tau(v-u)}{\varepsilon^2}\right\}\right) - \frac{\varepsilon^4 M}{32(v-u)^4}\right) \\
&\leq \exp\!\left(-K \min\!\left\{1, \frac{\varepsilon^{2\mathfrak{d}}}{(16(v-u)(\tau+1)^{\tau}R^\tau)^{\mathfrak{d}}}\right\}\right) \\
&\qquad + 2\exp\!\left(\mathfrak{d}\ln\!\left(\max\!\left\{1, \frac{128(\tau+1)^{\tau}R^\tau(v-u)}{\varepsilon^2}\right\}\right) - \frac{\varepsilon^4 M}{32(v-u)^4}\right).
\end{aligned} \tag{257}$$

The proof of Corollary 4.6 is thus completed. ☐

Corollary 4.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $d \in \mathbb{N}$, $L, a, u \in \mathbb{R}$, $b \in (a, \infty)$, $v \in (u, \infty)$, $R \in [\max\{1, L, |a|, |b|, 2|u|, 2|v|\}, \infty)$, let $X_m \colon \Omega \to [a,b]^d$, $m \in \mathbb{N}$, be i.i.d. random variables, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$, let $\varphi \colon [a,b]^d \to [u,v]$ satisfy for all $x, y \in [a,b]^d$ that $|\varphi(x) - \varphi(y)| \leq L\|x-y\|$, let $\mathfrak{l}_\tau \in \mathbb{N}^\tau$, $\tau \in \mathbb{N}$, satisfy for all $\tau \in \mathbb{N} \cap [3,\infty)$ that $\mathfrak{l}_\tau = (d, \tau, \tau, \ldots, \tau, 1)$, let $\mathfrak{E}_{\mathfrak{d},M,\tau} \colon [-R,R]^{\mathfrak{d}} \times \Omega \to [0,\infty)$, $\mathfrak{d}, M, \tau \in \mathbb{N}$, satisfy for all $\mathfrak{d}, M \in \mathbb{N}$, $\tau \in \mathbb{N} \cap [3,\infty)$, $\theta \in [-R,R]^{\mathfrak{d}}$, $\omega \in \Omega$ with $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ that*

$$\mathfrak{E}_{\mathfrak{d},M,\tau}(\theta,\omega) = \frac{1}{M}\left[\sum_{m=1}^M |\mathscr{N}_{u,v}^{\theta,\mathfrak{l}_\tau}(X_m(\omega)) - \varphi(X_m(\omega))|^2\right], \tag{258}$$

for every $\mathfrak{d} \in \mathbb{N}$ let $\Theta_{\mathfrak{d},k} \colon \Omega \to [-R,R]^{\mathfrak{d}}$, $k \in \mathbb{N}$, be i.i.d. random variables, assume for all $\mathfrak{d} \in \mathbb{N}$ that $\Theta_{\mathfrak{d},1}$ is continuous uniformly distributed on $[-R,R]^{\mathfrak{d}}$, and let $\Xi_{\mathfrak{d},K,M,\tau} \colon \Omega \to [-R,R]^{\mathfrak{d}}$, $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, satisfy for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$ that $\Xi_{\mathfrak{d},K,M,\tau} = \Theta_{\mathfrak{d},\min\{k \in \{1,2,\ldots,K\} \colon \mathfrak{E}_{\mathfrak{d},M,\tau}(\Theta_{\mathfrak{d},k}) = \min_{l \in \{1,2,\ldots,K\}} \mathfrak{E}_{\mathfrak{d},M,\tau}(\Theta_{\mathfrak{d},l})\}}$ (cf. Definition 2.8). Then there exists $c \in (0,\infty)$ such that for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, $\varepsilon \in (0, \sqrt{v-u}]$ with $\tau \geq 2d(2dL(b-a)\varepsilon^{-1} + 2)^d$ and $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ it holds that

$$\begin{aligned}
&\mathbb{P}\Bigg(\bigg[\int_{[a,b]^d} |\mathscr{N}_{u,v}^{\Xi_{\mathfrak{d},K,M,\tau},\mathfrak{l}_\tau}(x) - \varphi(x)|^2 \,\mathbb{P}_{X_1}(dx)\bigg]^{1/2} > \varepsilon\Bigg) \\
&\leq \exp\!\big(-K(c\tau)^{-\tau\mathfrak{d}}\varepsilon^{2\mathfrak{d}}\big) + 2\exp\!\big(\mathfrak{d}\ln\!\big((c\tau)^\tau\varepsilon^{-2}\big) - c^{-1}\varepsilon^4 M\big).
\end{aligned} \tag{259}$$

Proof of Corollary 4.7. Throughout this proof let $c \in (0,\infty)$ satisfy

$$c = \max\{32(v-u)^4, 256(v-u+1)R\}. \tag{260}$$

Note that Corollary 4.6 establishes that for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, $\varepsilon \in (0,\infty)$ with $\tau \geq 2d(2dL(b-a)\varepsilon^{-1}+2)^d$ and $\mathfrak{d} \geq \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ it holds that

$$\begin{aligned}
&\mathbb{P}\Bigg(\bigg[\int_{[a,b]^d} |\mathscr{N}_{u,v}^{\Xi_{\mathfrak{d},K,M,\tau},\mathfrak{l}_\tau}(x) - \varphi(x)|^2 \,\mathbb{P}_{X_1}(dx)\bigg]^{1/2} > \varepsilon\Bigg) \leq \exp\!\left(-K \min\!\left\{1, \frac{\varepsilon^{2\mathfrak{d}}}{(16(v-u)(\tau+1)^{\tau}R^\tau)^{\mathfrak{d}}}\right\}\right) \\
&\qquad + 2\exp\!\left(\mathfrak{d}\ln\!\left(\max\!\left\{1, \frac{128(\tau+1)^{\tau}R^\tau(v-u)}{\varepsilon^2}\right\}\right) - \frac{\varepsilon^4 M}{32(v-u)^4}\right).
\end{aligned} \tag{261}$$

Next observe that (260) ensures that for all $\tau \in \mathbb{N}$ it holds that

$$16(v-u)(\tau+1)^\tau R^\tau \leq (16(v-u+1)(\tau+1)R)^\tau \leq (32(v-u+1)R\tau)^\tau \leq (c\tau)^\tau. \tag{262}$$

The fact that for all $\varepsilon \in (0, \sqrt{v-u}]$, $\tau \in \mathbb{N}$ it holds that $\varepsilon^2 \le 16(v-u)(\tau+1)^\tau R^\tau$ therefore shows that for all $\varepsilon \in (0, \sqrt{v-u}]$, $\tau \in \mathbb{N}$ it holds that

$$-\min\left\{1, \frac{\varepsilon^{2\mathfrak{d}}}{(16(v-u)(\tau+1)^\tau R^\tau)^{\mathfrak{d}}}\right\} = \frac{-\varepsilon^{2\mathfrak{d}}}{(16(v-u)(\tau+1)^\tau R^\tau)^{\mathfrak{d}}} \le \frac{-\varepsilon^{2\mathfrak{d}}}{(c\tau)^{\tau\mathfrak{d}}}. \tag{263}$$

Furthermore, note that (260) implies that for all $\tau \in \mathbb{N}$ it holds that

$$128(\tau+1)^\tau R^\tau(v-u) \le 128(2\tau)^\tau R^\tau(v-u) \le (256R\tau(v-u+1))^\tau \le (c\tau)^\tau. \tag{264}$$

The fact that for all $\varepsilon \in (0, \sqrt{v-u}]$, $\tau \in \mathbb{N}$ it holds that $\varepsilon^2 \le 128(\tau+1)^\tau R^\tau(v-u)$ hence proves that for all $\varepsilon \in (0, \sqrt{v-u}]$, $\tau \in \mathbb{N}$ it holds that

$$\ln\left(\max\left\{1, \frac{128(\tau+1)^\tau R^\tau(v-u)}{\varepsilon^2}\right\}\right) = \ln\left(\frac{128(\tau+1)^\tau R^\tau(v-u)}{\varepsilon^2}\right) \le \ln\left(\frac{(c\tau)^\tau}{\varepsilon^2}\right) \tag{265}$$

In addition, observe that (260) ensures that

$$\frac{-1}{32(v-u)^4} \le \frac{-1}{c}. \tag{266}$$

Combining this, (263), and (265) with (261) proves that for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, $\varepsilon \in (0, \sqrt{v-u}]$ with $\tau \ge 2d(2dL(b-a)\varepsilon^{-1}+2)^d$ and $\mathfrak{d} \ge \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ it holds that

$$\begin{aligned}&\mathbb{P}\left(\left[\int_{[u,v]^d} |\mathscr{N}_{u,v}^{\Xi_{\mathfrak{d},K,M,\tau},\mathfrak{l}_\tau}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx)\right]^{1/2} > \varepsilon\right) \\ &\le \exp\left(\frac{-K\varepsilon^{2\mathfrak{d}}}{(c\tau)^{\tau\mathfrak{d}}}\right) + 2\exp\left(\mathfrak{d}\ln\left(\frac{(c\tau)^\tau}{\varepsilon^2}\right) - \frac{\varepsilon^4 M}{c}\right).\end{aligned} \tag{267}$$

The proof of Corollary 4.7 is thus completed. □

Corollary 4.8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $d \in \mathbb{N}$, $L, a, u \in \mathbb{R}$, $b \in (a, \infty)$, $v \in (u, \infty)$, $R \in [\max\{1, L, |a|, |b|, 2|u|, 2|v|\}, \infty)$, let $X_m \colon \Omega \to [a,b]^d$, $m \in \mathbb{N}$, be i.i.d. random variables, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$, let $\varphi \colon [a,b]^d \to [u,v]$ satisfy for all $x, y \in [a,b]^d$ that $|\varphi(x) - \varphi(y)| \le L\|x-y\|$, let $\mathfrak{l}_\tau \in \mathbb{N}^\tau$, $\tau \in \mathbb{N}$, satisfy for all $\tau \in \mathbb{N} \cap [3,\infty)$ that $\mathfrak{l}_\tau = (d, \tau, \tau, \ldots, \tau, 1)$, let $\mathfrak{E}_{\mathfrak{d},M,\tau} \colon [-R,R]^{\mathfrak{d}} \times \Omega \to [0,\infty)$, $\mathfrak{d}, M, \tau \in \mathbb{N}$, satisfy for all $\mathfrak{d}, M \in \mathbb{N}$, $\tau \in \mathbb{N} \cap [3,\infty)$, $\theta \in [-R,R]^{\mathfrak{d}}$, $\omega \in \Omega$ with $\mathfrak{d} \ge \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ that*

$$\mathfrak{E}_{\mathfrak{d},M,\tau}(\theta,\omega) = \frac{1}{M}\left[\sum_{m=1}^{M} |\mathscr{N}_{u,v}^{\theta,\mathfrak{l}_\tau}(X_m(\omega)) - \varphi(X_m(\omega))|^2\right], \tag{268}$$

for every $\mathfrak{d} \in \mathbb{N}$ let $\Theta_{\mathfrak{d},k} \colon \Omega \to [-R,R]^{\mathfrak{d}}$, $k \in \mathbb{N}$, be i.i.d. random variables, assume for all $\mathfrak{d} \in \mathbb{N}$ that $\Theta_{\mathfrak{d},1}$ is continuous uniformly distributed on $[-R,R]^{\mathfrak{d}}$, and let $\Xi_{\mathfrak{d},K,M,\tau} \colon \Omega \to [-R,R]^{\mathfrak{d}}$, $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, satisfy for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$ that $\Xi_{\mathfrak{d},K,M,\tau} = \Theta_{\mathfrak{d},\min\{k \in \{1,2,\ldots,K\} \colon \mathfrak{E}_{\mathfrak{d},M,\tau}(\Theta_{\mathfrak{d},k}) = \min_{l \in \{1,2,\ldots,K\}} \mathfrak{E}_{\mathfrak{d},M,\tau}(\Theta_{\mathfrak{d},l})\}}$ (cf. Definition 2.8). Then there exists $c \in (0,\infty)$ such that for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, $\varepsilon \in (0, \sqrt{v-u}]$ with $\tau \ge 2d(2dL(b-a)\varepsilon^{-1}+2)^d$ and $\mathfrak{d} \ge \tau(d+1) + (\tau-3)\tau(\tau+1) + \tau + 1$ it holds that

$$\begin{aligned}&\mathbb{P}\left(\int_{[a,b]^d} |\mathscr{N}_{u,v}^{\Xi_{\mathfrak{d},K,M,\tau},\mathfrak{l}_\tau}(x) - \varphi(x)| \, \mathbb{P}_{X_1}(dx) > \varepsilon\right) \\ &\le \exp\big(-K(c\tau)^{-\tau\mathfrak{d}}\varepsilon^{2\mathfrak{d}}\big) + 2\exp\big(\mathfrak{d}\ln\big((c\tau)^\tau\varepsilon^{-2}\big) - c^{-1}\varepsilon^4 M\big).\end{aligned} \tag{269}$$

Proof of Corollary 4.8. Note that Jensen's inequality shows that for all $f \in C([a,b]^d, \mathbb{R})$ it holds that

$$\int_{[a,b]^d} |f(x)| \, \mathbb{P}_{X_1}(dx) \le \left[\int_{[a,b]^d} |f(x)|^2 \, \mathbb{P}_{X_1}(dx)\right]^{\frac{1}{2}}. \tag{270}$$

Combining this with Corollary 4.7 proves (269). The proof of Corollary 4.8 is thus completed. □

4.3.2 Convergence rates for strong convergence

Lemma 4.9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $c \in [0, \infty)$, and let $X: \Omega \to [-c, c]$ be a random variable. Then it holds for all $\varepsilon, p \in (0, \infty)$ that*

$$\mathbb{E}[|X|^p] \leq \varepsilon^p \, \mathbb{P}(|X| \leq \varepsilon) + c^p \, \mathbb{P}(|X| > \varepsilon) \leq \varepsilon^p + c^p \, \mathbb{P}(|X| > \varepsilon). \tag{271}$$

Proof of Lemma 4.9. Observe that the hypothesis that for all $\omega \in \Omega$ it holds that $|X(\omega)| \leq c$ ensures that for all $\varepsilon, p \in (0, \infty)$ it holds that

$$\mathbb{E}[|X|^p] = \mathbb{E}\big[|X|^p \mathbb{1}_{\{|X| \leq \varepsilon\}}\big] + \mathbb{E}\big[|X|^p \mathbb{1}_{\{|X| > \varepsilon\}}\big] \leq \varepsilon^p \, \mathbb{P}(|X| \leq \varepsilon) + c^p \, \mathbb{P}(|X| > \varepsilon) \leq \varepsilon^p + c^p \, \mathbb{P}(|X| > \varepsilon). \tag{272}$$

The proof of Lemma 4.9 is thus completed. □

Corollary 4.10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $d \in \mathbb{N}$, $L, a, u \in \mathbb{R}$, $b \in (a, \infty)$, $v \in (u, \infty)$, $R \in [\max\{1, L, |a|, |b|, 2|u|, 2|v|\}, \infty)$, let $X_m: \Omega \to [a, b]^d$, $m \in \mathbb{N}$, be i.i.d. random variables, let $\|\cdot\|: \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $\varphi: [a, b]^d \to [u, v]$ satisfy for all $x, y \in [a, b]^d$ that $|\varphi(x) - \varphi(y)| \leq L\|x - y\|$, let $\mathfrak{l}_\tau \in \mathbb{N}^\tau$, $\tau \in \mathbb{N}$, satisfy for all $\tau \in \mathbb{N} \cap [3, \infty)$ that $\mathfrak{l}_\tau = (d, \tau, \tau, \ldots, \tau, 1)$, let $\mathfrak{E}_{\mathfrak{d}, M, \tau}: [-R, R]^{\mathfrak{d}} \times \Omega \to [0, \infty)$, $\mathfrak{d}, M, \tau \in \mathbb{N}$, satisfy for all $\mathfrak{d}, M \in \mathbb{N}$, $\tau \in \mathbb{N} \cap [3, \infty)$, $\theta \in [-R, R]^{\mathfrak{d}}$, $\omega \in \Omega$ with $\mathfrak{d} \geq \tau(d + 1) + (\tau - 3)\tau(\tau + 1) + \tau + 1$ that*

$$\mathfrak{E}_{\mathfrak{d}, M, \tau}(\theta, \omega) = \frac{1}{M} \left[\sum_{m=1}^{M} |\mathscr{N}_{u,v}^{\theta, \mathfrak{l}_\tau}(X_m(\omega)) - \varphi(X_m(\omega))|^2 \right], \tag{273}$$

for every $\mathfrak{d} \in \mathbb{N}$ let $\Theta_{\mathfrak{d}, k}: \Omega \to [-R, R]^{\mathfrak{d}}$, $k \in \mathbb{N}$, be i.i.d. random variables, assume for all $\mathfrak{d} \in \mathbb{N}$ that $\Theta_{\mathfrak{d}, 1}$ is continuous uniformly distributed on $[-R, R]^{\mathfrak{d}}$, and let $\Xi_{\mathfrak{d}, K, M, \tau}: \Omega \to [-R, R]^{\mathfrak{d}}$, $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, satisfy for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$ that $\Xi_{\mathfrak{d}, K, M, \tau} = \Theta_{\mathfrak{d}, \min\{k \in \{1, 2, \ldots, K\}: \, \mathfrak{E}_{\mathfrak{d}, M, \tau}(\Theta_{\mathfrak{d}, k}) = \min_{l \in \{1, 2, \ldots, K\}} \mathfrak{E}_{\mathfrak{d}, M, \tau}(\Theta_{\mathfrak{d}, l})\}}$ (cf. Definition 2.8). Then there exists $c \in (0, \infty)$ such that for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, $p \in [1, \infty)$, $\varepsilon \in (0, \sqrt{v - u}]$ with $\tau \geq 2d(2dL(b - a)\varepsilon^{-1} + 2)^d$ and $\mathfrak{d} \geq \tau(d + 1) + (\tau - 3)\tau(\tau + 1) + \tau + 1$ it holds that

$$\begin{aligned} &\left(\mathbb{E}\left[\left(\int_{[a,b]^d} |\mathscr{N}_{u,v}^{\Xi_{\mathfrak{d}, K, M, \tau}, \mathfrak{l}_\tau}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) \right)^{p/2} \right] \right)^{1/p} \\ &\leq (v - u) \left[\exp\big(-K(c\tau)^{-\tau\mathfrak{d}} \varepsilon^{2\mathfrak{d}}\big) + 2 \exp\big(\mathfrak{d} \ln\big((c\tau)^\tau \varepsilon^{-2}\big) - c^{-1}\varepsilon^4 M\big) \right]^{1/p} + \varepsilon. \end{aligned} \tag{274}$$

Proof of Corollary 4.10. First, observe that Corollary 4.7 ensures that there exists $c \in (0, \infty)$ which satisfies for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, $\varepsilon \in (0, \sqrt{v - u}]$ with $\tau \geq 2d(2dL(b - a)\varepsilon^{-1} + 2)^d$ and $\mathfrak{d} \geq \tau(d + 1) + (\tau - 3)\tau(\tau + 1) + \tau + 1$ that

$$\begin{aligned} &\mathbb{P}\left(\left[\int_{[a,b]^d} |\mathscr{N}_{u,v}^{\Xi_{\mathfrak{d}, K, M, \tau}, \mathfrak{l}_\tau}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) \right]^{1/2} > \varepsilon \right) \\ &\leq \exp\big(-K(c\tau)^{-\tau\mathfrak{d}} \varepsilon^{2\mathfrak{d}}\big) + 2 \exp\big(\mathfrak{d} \ln\big((c\tau)^\tau \varepsilon^{-2}\big) - c^{-1}\varepsilon^4 M\big). \end{aligned} \tag{275}$$

Lemma 4.9 (with $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $c \leftarrow v - u$, $X \leftarrow (\Omega \ni \omega \mapsto \big[\int_{[a,b]^d} |\mathscr{N}_{u,v}^{\Xi_{\mathfrak{d}, K, M, \tau}(\omega), \mathfrak{l}_\tau}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx)\big]^{1/2} \in [u - v, v - u])$ in the notation of Lemma 4.9) hence ensures that for all $\mathfrak{d}, K, M, \tau \in \mathbb{N}$, $\varepsilon \in (0, \sqrt{v - u}]$, $p \in (0, \infty)$ with $\tau \geq 2d(2dL(b - a)\varepsilon^{-1} + 2)^d$ and $\mathfrak{d} \geq \tau(d + 1) + (\tau - 3)\tau(\tau + 1) + \tau + 1$ it holds that

$$\begin{aligned} &\mathbb{E}\left[\left(\int_{[a,b]^d} |\mathscr{N}_{u,v}^{\Xi_{\mathfrak{d}, K, M, \tau}, \mathfrak{l}_\tau}(x) - \varphi(x)|^2 \, \mathbb{P}_{X_1}(dx) \right)^{p/2} \right] \\ &\leq \varepsilon^p + (v - u)^p \left[\exp\big(-K(c\tau)^{-\tau\mathfrak{d}} \varepsilon^{2\mathfrak{d}}\big) + 2 \exp\big(\mathfrak{d} \ln\big((c\tau)^\tau \varepsilon^{-2}\big) - c^{-1}\varepsilon^4 M\big) \right]. \end{aligned} \tag{276}$$

The fact that for all $p \in [1, \infty)$, $x, y \in [0, \infty)$ it holds that $(x + y)^{1/p} \leq x^{1/p} + y^{1/p}$ therefore establishes (274). The proof of Corollary 4.10 is thus completed. □

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