Specifications for writing a thesis or a scientific article

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Contents

1 Preface

The following document provides detailed specifications regarding the way in which a thesis or a scientific article should be written. In principle, all specifications are up for discussion. If you intend to deviate from one of the specifications below, then please contact me *in advance* before deviating from the proposed specifications to discuss with me whether I agree to the deviations that you propose. If you do not contact me regarding the proposed specifications, then I assume that you fully agree with the proposed specifications. Please let me know if you have any questions regarding this document. Please also study carefully the LaTeX code used to create this document!

2 General comments

2.1 Logical precision

1. *Each predicate* should be quantified by the use of a quantifier (often in written text instead of with a symbol) *in front* of the predicate. The most common quantifiers are the *universal* quantifier (text: "for all," symbol: $\forall$) and the *existential* quantifier (text: "there exist(s)," symbol: $\exists$). The word "For" is not a quantifier.

   For example, please write: note that for all $x \in \mathbb{R}$ there exists a real number $y \in \mathbb{R}$ such that for all $t \in [0, \infty)$ it holds that

   $$e^x \le y\, e^t \tag{2.1}$$

   instead of, for example:

   - note that for all $x \in \mathbb{R}$ there exists a real number $y \in \mathbb{R}$ such that

     $$e^x \le y\, e^t \tag{2.2}$$

     for all $t \in [0, \infty)$ (Problem: it is not entirely clear whether Eq. (2.2) means "$\forall\, x \in \mathbb{R} \colon \exists\, y \in \mathbb{R} \colon \big(e^x \le y\, e^t \;\forall\, t \in [0, \infty)\big)$" or whether it means "$\forall\, x \in \mathbb{R} \colon \big(\big(\exists\, y \in \mathbb{R} \colon e^x \le y\, e^t\big)\, \forall\, t \in [0, \infty)\big)$". Putting the quantifiers in front of the predicate avoids such issues.),

   - note that for all $x \in \mathbb{R}$ there exists a real number $y \in \mathbb{R}$ such that for $t \in [0, \infty)$:

     $$e^x \le y\, e^t \tag{2.3}$$

     (Problem: it is not clear what "for $t \in [0, \infty)$" means. Does it mean "for all $t \in [0, \infty)$ it holds that" or does it mean "for some $t \in [0, \infty)$ it holds that"? "For some $t \in [0, \infty)$ it holds that," in turn, is another way of saying that "there exists a $t \in [0, \infty)$ such that."), or

   - note that for all $x \in \mathbb{R}$ there exists a real number $y \in \mathbb{R}$ such that

     $$e^x \le y\, e^t \tag{2.4}$$

     (Problem: t is not quantified in Eq. (2.4); that is, Eq. (2.4) is not a statement but an 1-ary predicate. So, it does not make sense to say that we should note that Eq. (2.4) holds.).

   Wikipedia (German) http://de.wikipedia.org/wiki/Pr%C3%A4dikatenlogik_erster_Stufe (access date: October 8, 2021):

"The first-order logic is a branch of mathematical logic. It is concerned with the structure of certain mathematical expressions and the logical inference that goes from such expressions to others. In doing so, it is possible to define both the language and the inference in a purely syntactic way, i.e., without reference to mathematical meanings. The interplay of purely syntactic considerations on the one hand and semantic considerations on the other hand leads to important findings that are important for all of mathematics, because this can be formulated using the Zermelo-Fraenkel set theory in the first-level predicate logic."

2. Please also use "Let $n \in [0, \infty)$" or "Let $n \in \{0, 1, 2, \dots\}$" (depending on what you mean) instead of "Let $n \geq 0$" as it is not entirely clear whether "Let $n \geq 0$" means "Let $n \in \{0, 1, 2, \dots\}$" or "Let $n \in [0, \infty)$."

3. We never use the word "define" and we never use the expression ":=" in the article/ thesis. Instead we write, for example,

   (i) let A be the set given by $A = \{2, 4, 6, 8, \dots\}$,

   (ii) let A_r, $r \in \mathbb{R}$, be the sets which satisfy for all $r \in \mathbb{R}$ that $A_r = [r, \infty)$, or

   (iii) let $f_r \colon \mathbb{R} \to \mathbb{R}$, $r \in \mathbb{R}\backslash\{0\}$, be the functions which satisfy for all $r \in \mathbb{R}\backslash\{0\}$, $x \in \mathbb{R}$ that
$$f_r(x) = \frac{1}{\sqrt{2\pi r^2}} \exp\!\left(-\frac{x^2}{2r^2}\right). \tag{2.5}$$

2.2 Writing style and scientific English

(i) The LaTeX package `hyperref` must be included into the document.

(ii) Every time after the symbol ∀ or the symbol ∃ has appeared there must appear the expression `\,` in the LaTeX code.

(iii) The following LaTeX phrases may not appear in your document:

   (a) "`cf. `"

   (b) "`i.i.d. `"

   Instead the following LaTeX phrases may appear in your document:

   (a) "`cf.\ `"

   (b) "`i.i.d.\ `"

   Analogous comments apply to other abbreviations.

(iv) The following LaTeX phrases may not appear in your document:

   (a) "`$ d, m \in \N, x \in \R $`"

   (b) "`$ d_1, d_2, \dots, d_k \in \N, x \in \R $`"

   Instead the following LaTeX phrases may appear in your document:

(a) “`$ d, m \in \N $, $ x \in \R $`”

(b) “`$ d_1, d_2, \dots, d_k \in \N $, $ x \in \R $`”

Analogous comments apply to similar expressions.

(v) Immediately after the phrase “e.g.,” immediately after the phrase “for example,” and immediately after the phrase “for instance” must follow a comma.

(vi) One should never write that a function “is increasing.” Instead one should write that a function “is strictly increasing” or “non-decreasing” (depending on the correct content). The analogous comment applies to the phrase that a function “is decreasing.”

(vii) The expressions $\subset$, $\mathbb{R}^+$, $\mathbb{R}_{\geq 0}$, or $\mathbb{R}^*$ may nowhere appear in the document. Instead the expressions $\subseteq$, $\subsetneq$, $(0, \infty)$, $[0, \infty)$, or $\mathbb{R}\backslash\{0\}$ can be used in the document.

(viii) Immediately after every result (by which we mean a theorem, a lemma, a proposition, or a corollary) of the document (by which we mean a scientific article or a thesis)

(i) we add a proof which starts with “Proof of . . . ” and which ends, e.g., with “The proof of . . . is thus complete.” or

(ii) we add a reference to another article/thesis for the proof of the result in front of the result.

See the proof of Proposition 4.1 for an example.

(ix) Typically, every sentence, except for the last one (and sometimes also the first one) in the proof of a theorem, a lemma, a proposition, or a corollary starts with something like

(a) This implies/shows/proves/ensures/assures/yields/demonstrates that . . .

(b) This, equation (...), inequality (...), estimate (...), Theorem ..., Corollary ..., Lemma ..., and Lemma ... imply/show/prove/ensure/assure/yield/demonstrate that . . .

(c) Combining . . . (with . . .) (hence/therefore) implies/shows/. . . that . . .

(d) Theorem ..., Corollary ..., Corollary ..., and Proposition ... (hence/therefore) imply/show/. . . that . . .

Typically, the last sentence in the proof reads as “The proof of . . . is thus complete.”. Throughout the entire article/thesis the words “thus” typically appears only in the last sentence of a proof. A proof starts often with “Throughout this proof let . . . , let . . . , . . . , and let . . . . Next observe/note that . . . ”.

(x) Just before every display (LaTeX `\begin{equation} ... \end{equation}`) in the document should appear the phrase “that”, the phrase “and”, the phrase “Then”, or the phrase “be the set given by” without any punctuation mark.

For example, we often write

This implies for all … that
`\begin{equation} ... \end{equation}`

or

This implies that for all … it holds that
`\begin{equation} ... \end{equation}`

in a proof.

(xi) The phrase "and that" may not appear in any result nor in any proof.

(xii) Typically, we do not write a long text between the results. One sentence relating the result with other results/notions/concepts in the document as well as with other results/notions/concepts in the literature is often sufficient. We may add more explanatory sentences (written text) in the introduction and in the beginning of the section/chapter.

(xiii) The document should start with an abstract and, thereafter, a table of contents.

(xiv) Typically, all assumptions in a setting are presented in exactly one sentence. In particular, we typically formulate a setting as follows:

Throughout this article the following setting is (frequently[1]) used. Assume/let …, assume/let …, assume/let …, …, and assume/let … .

(xv) Every display has exactly one number on the right hand side. In particular, we never use `\begin{equation*} ... \end{equation*}` or `$$ ... $$`.
We always use
`\begin{equation} ... \end{equation}`
or
`\begin{equation} \begin{split} ... \end{split} \end{equation}`
instead; see Eq. (4.1)–Eq. (4.8) for a few examples.

(xvi) We never use

Assumption 2.1. Throughout this article/thesis assume that $Tp < 1$, that for all $x \in \mathbb{R}$ it holds that

$$f(x) \leq 2x, \tag{2.6}$$

and that for all $y \in [0, \infty)$ it holds that

$$g(y) \leq 2y. \tag{2.7}$$

but we may use one or more settings (see item (xi) above and see Section 4.1.2 and Proposition 4.3 for an example).

(xvii) Often the structure of a thesis/research article is as follows:

(a) Introduction

[1] depending upon whether the setting is always or only sometimes used throughout the document

1.1 Notation

1.2 Setting

(b) First section for results ...

(c) Second section for results ...

(d) ...

(xviii) Serial comma (also called Oxford comma): We write "A, B, and C" instead of "A, B and C". Similarly we write "A, B, or C" instead of "A, B or C."

(a) However, please note that the phrase "let $f\colon \mathbb{R} \to \mathbb{R}$, and $g\colon \mathbb{R} \to \mathbb{R}$ be functions which satisfy" may not appear in your document.

(b) Instead the phrase "let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions which satisfy" may appear in your document.

(xix) There should be no significant LaTeX badboxes in the document.

(xx) You are welcome to use the axiom schema of specification to specify a set. More specifically, if S is a set and P is a predicate with one single argument, then you can use the set

$$\{x \in S\colon P(x)\} \tag{2.8}$$

according to the axiom schema of specification in your document. In contrast, the expression

$$\{x \in S \,|\, P(x)\} \tag{2.9}$$

may not appear in your document. Moreover, the expression

$$\{x \in S : P(x)\} \tag{2.10}$$

may also not appear in your document. As usual please check the employed LaTeX code for Eq. (2.8) and Eq. (2.10), respectively.

(xxi) You are welcome to employ to the LaTeX command `\begin{cases} ... \end{cases}` to create a distinction of cases within a display. If you employ the LaTeX command `\begin{cases} ... \end{cases}`, then it must employ the following structure:

```
\begin{cases}
Value in the first case
&
\colon
First case
\\
Value in the second case
&
\colon
Second case
```

```
\\
Value in the third case
&
\colon
Third case
\\
...
\\
Value in the n-th case
&
\colon
n-th case
\end{cases}
```

For example, you may use the following phrase "let $a \in \mathbb{R}$, let $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ be the function which satisfies for all $x \in \mathbb{R}$ that

$$\operatorname{sgn}(x) = \begin{cases} 1 & \colon x \geq 0 \\ -1 & \colon x < 0, \end{cases} \tag{2.11}$$

and let ..." to introduce a real number $a \in \mathbb{R}$ and to specify the sign function (a sign function). The phrase "let $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ be the function that satisfies" may not appear in your document.

(xxii) The phrase "let $a \in \mathbb{R}$, let $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ be the function which satisfies" may not appear in your document. The phrase "let $a \in \mathbb{R}$, let $sgn\colon \mathbb{R} \to \mathbb{R}$ be the function which satisfies" may also not appear in your document. Please note the differences between $\operatorname{sgn}\colon \mathbb{R} \to \mathbb{R}$ (cf. Eq. (2.11) in item (xxi) above), $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ (see above), and $sgn\colon \mathbb{R} \to \mathbb{R}$ (see above) and as usual please also study the LaTeX codes used to create the above phrases.

(xxiii) The phrase "(c.f. Eq. (2.11))" may not appear in your document. The phrase "(cf. Eq. (2.11)" may also not appear in your document. In contrast, the phrase "(cf. Eq. (2.11))" and the phrase "(see Eq. (2.11))" may appear in your document. As usual please also study the LaTeX codes used to create the above phrases.

(xxiv) The phrase "let $\|\cdot\|_{\mathbb{R}^d}\colon \mathbb{R}^d \to [0, \infty)$ be the Euclidean norm on $\mathbb{R}^d$" may appear in your document.

(a) The phrase "let $\|\cdot\|_{\mathbb{R}^d}\colon \mathbb{R}^d \to [0, \infty)$ be the euclidean norm on $\mathbb{R}^d$" may not appear in your document. Analogously, one writes "is Hölder continuous" instead of is "hölder continuous."

(b) The phrase "let $\| \cdot \|_{\mathbb{R}^d}\colon \mathbb{R}^d \to [0, \infty)$ be the Euclidean norm on $\mathbb{R}^d$" may also not appear in your document.

As usual please also study the LaTeX codes used to create the above phrases.

(xxv) The phrase "function" and the phrase "functions" may appear in your document. In contrast, each of the following phrases may not appear in your document:

(a) mapping
(b) Mapping
(c) map
(d) Map
(e) Function
(f) Functions

The following phrases may appear in your document:

(a) let $f\colon \mathbb{R} \to \mathbb{R}$ be the function which satisfies for all
(b) let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy for all
(c) let $g_r\colon \mathbb{R} \to \mathbb{R}$, $r \in (0, \infty)$, be the functions which satisfy for all $r \in (0, \infty)$, $x \in \mathbb{R}$ that $g_r(x) = x + r$
(d) let $g_r\colon \mathbb{R} \to \mathbb{R}$, $r \in (0, \infty)$, satisfy for all $r \in (0, \infty)$, $x \in \mathbb{R}$ that $g_r(x) = x + r$
(e) let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions
(f) let $f = (f(x, y))_{(x,y)\in\mathbb{R}^2}\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions
(g) let $f = (f_x(y))_{(x,y)\in\mathbb{R}^2}\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions
(h) let $g_r \in C(\mathbb{R}, \mathbb{R})$, $r \in (0, \infty)$, satisfy
(i) let $a_n \in \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$ that $a_n = n$
(j) let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ that $a_n = n$

In contrast, the following phrases may not appear in your document:

(a) let $f : \mathbb{R} \to \mathbb{R}$ be the function which satisfies for all
(b) let $g_r\colon \mathbb{R} \to \mathbb{R}$, $r \in (0, \infty)$ be the functions which satisfy for all $r \in (0, \infty)$, $x \in \mathbb{R}$ that $g_r(x) = x + r$
(c) let $g_r\colon \mathbb{R} \to \mathbb{R}$, $r \in (0, \infty)$, be the functions which satisfy for all $x \in \mathbb{R}$ that $g_r(x) = x + r$
(d) let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$ be functions
(e) let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$, be functions
(f) let $g_r \in C(\mathbb{R}, \mathbb{R})$, $r \in (0, \infty)$ satisfy
(g) Let $g_r \in C(\mathbb{R}, \mathbb{R})$ satisfy
(h) Let $a \in \mathbb{R}$ and let $g_r \in C(\mathbb{R}, \mathbb{R})$ satisfy
(i) let $g_r \in C(\mathbb{R}, \mathbb{R}), r \in (0, \infty)$, satisfy
(j) let $g_r \in C(\mathbb{R}, \mathbb{R}), r \in (0, \infty)$, satisfy

(k) let $a_n \in \mathbb{R}$, $n \in \mathbb{N}$, satisfy that $a_n = n$

(l) let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy that $a_n = n$

(m) let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$ that $a_n = n$

(n) let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, $n \in \mathbb{N}$ satisfy for all $n \in \mathbb{N}$ that $a_n = n$

As usual please also study the LaTeX codes used to create the above phrases.

(A) In your document you may use the following phrases:

  (a) let $f : \mathbb{R} \to \mathbb{R}$ be the function which satisfies for all

  (b) let $g_r : \mathbb{R} \to \mathbb{R}$, $r \in (0, \infty)$, be the functions which satisfy for all $r \in (0, \infty)$, $x \in \mathbb{R}$ that $g_r(x) = x + r$

  (c) let $g_r \in C(\mathbb{R}, \mathbb{R})$, $r \in (0, \infty)$, satisfy

(B) Instead of the phrases in item (A) you may also use the following phrases:

  (a) let $f : \mathbb{R} \to \mathbb{R}$ satisfy for all

  (b) let $g_r : \mathbb{R} \to \mathbb{R}$, $r \in (0, \infty)$, satisfy for all $r \in (0, \infty)$, $x \in \mathbb{R}$ that $g_r(x) = x + r$

  (c) let $g_r \in C(\mathbb{R}, \mathbb{R})$, $r \in (0, \infty)$, satisfy

You may, however, not mix between item (A) and item (B). Please use either the phrases in item (A) or the phrases in item (B).

(xxvi) Except in the introduction and the text between the presented results, the phrase "satisfying" and the phrase "fulfilling" may not appear in your document.

3 Precise specifications

3.1 Structure of the document

A document (by which we mean a scientific article or a thesis) consists of exactly the following parts (in this order):

1. Title of the document (Title)

2. Name of the authors

3. Affiliations of the authors

4. Abstract of the document (Abstract)

5. Table of contents of the document

6. Main text of the document (Main text)

7. List of references.

The main text contains of the following LaTeX sections:

1. Introduction
2. Name of the 2nd section
3. Name of the 3rd section
4. ...

3.2 LaTeX header

The LaTeX header should not import packages which are not used in the document. The following LaTeX code provides an illustrative example.

```
\documentclass[a4paper,12pt]{article}
\usepackage{amsthm,
            amsmath,
            amssymb,
            bbm,
            enumerate,
            geometry,
            nicefrac,
            hyperref
            }

\newtheorem{theorem}{Theorem}
\newtheorem{lemma}{Lemma}
\newtheorem{remark}{Remark}
\newtheorem{prop}{Proposition}
\newtheorem{cor}{Corollary}

\newcommand{\R}{\mathbb{R}}
\newcommand{\N}{\mathbb{N}}
\newcommand{\E}{\mathbb{E}}
\renewcommand{\P}{\mathbb{P}}
\newcommand{\tr}{\operatorname{trace}}

\title{Title of the document}

\author{First author$^1$, Second author$^2$, and Third Author$^{3}$
\bigskip
\\
\small{$^1$Department of Mathematics, Some University,
USA, e-mail: $ \dots @ \dots $}
\smallskip
\\
\small{$^2$Department of Mathematics, ETH Zurich,
```

```
Switzerland, e-mail: $ \dots @ \dots $}
\smallskip
\\
\small{$^3$Faculty of Mathematics, Bielefeld University,
Germany, e-mail: $ \dots @ \dots $}}

\begin{document}

\maketitle

\begin{abstract}
Abstract ...
\end{abstract}

\tableofcontents

\section{Introduction}

\bibliographystyle{acm}
\bibliography{bibfile}
\end{document}
```

3.3 Main text of the document

The main text consists of environments (LaTeX environments) with free text in between. The free text does not follow precise specifications. In Section 1 there is typically a lot of free text. In all other sections there is typically only very little free text. The purpose of the free text in all other sections beside Section 1 is typically to link the presented environment to other environments in the document as well as to link the presented environment to other findings in the scientific literature.

3.4 LaTeX environments

An environment is

1. a remark,
2. a setting,
3. a framework,
4. a definition,
5. a result (by which we mean a theorem, a lemma, a proposition, or a corollary), or
6. the proof of a result.

In LaTeX, an environment is formulated by means of the commands:

```
\begin{type_of_the_environment}
\label{...}
...
\end{type_of_the_environment}
```

It is not allowed to use the word *where*, the word *from*, the word *since*, the word *as*, the word *by*, the word *because*, or the word *using* in a LaTeX environment.

3.4.1 Remarks

A remark does not necessarily need to follow precise specifications. All other environments follow precise specifications.

3.4.2 Settings and frameworks

A setting and a framework consists of exactly one sentence. This sentence has the following structure (illustrated here in the case of a setting environment):

Setting 3.1. Assume/let ..., assume/let ..., assume/let ..., ..., assume/let ..., and assume/let ... .

A framework environment is sometimes useful to describe algorithms. In most cases the document under consideration does not contain any framework but there might be several settings.

3.4.3 Definitions

Every definition either consists of exactly one sentence or consists of exactly two sentences.

Definitions consisting of exactly one sentence If a definition consists of exactly one sentence, then the definition must follow the following structure:

Definition 3.2. We denote by ... .

Definitions consisting of exactly two sentences If a definition consists of exactly two sentences, then the definition must follow the following structure:

Definition 3.3. Let ..., let ..., ..., and let ... Then we denote by ... .

3.4.4 Results

If a result consists of more than one sentence, then the second sentence of the result must start with the word "Then."

Results consisting of exactly one sentence If a result consists of exactly one sentence, then the result must follow one of the following two structures (illustrated here in the case of a theorem environment):

1. First option for a result (theorem, lemma, proposition, or corollary) with exactly one sentence (illustrated here in the case of a theorem environment):

   Theorem 3.4. It holds for all ... that ....

2. Second option for a result (theorem, lemma, proposition, or corollary) with exactly one sentence (illustrated here in the case of a theorem environment):

   Theorem 3.5. There exist(s) ... that ....

Results consisting of exactly two sentences In most cases a result consists of exactly two sentences. In this case the result (theorem, lemma, proposition, or corollary) must follow the following structure (illustrated here in the case of a theorem environment):

Theorem 3.6. Assume/let ..., assume/let ..., assume/let ..., ..., assume/let ..., and assume/let .... Then ....

The first sentence in Theorem 3.6 formulates the employed hypotheses. If a setting is employed as one hypothesis, then one writes (where $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ is a natural number): *Assume Setting ??* (1st hypothesis), *assume/let* ... (2nd hypothesis), *assume/let* ... (3rd hypothesis), ..., *assume/let* ... ($(n-1)$-th hypothesis), *and assume/let* ... (n-th hypothesis). In most cases the second sentence of a result starts

(i) with "Then it holds for all ..." or

(ii) with "Then there exist(s) ...."

Results consisting of more than two sentences If a result (theorem, lemma, proposition, or corollary) consists of more than two sentences, then the result must follow the following structure (illustrated here in the case of a theorem environment):

Theorem 3.7. Assume/let ..., assume/let ..., assume/let ..., ..., assume/let ..., and assume/let .... Then the following two/three/four/five/... statements are equivalent:

(i) It holds .../There exist(s) ....

(ii) It holds .../There exist(s) ....

(iii) It holds .../There exist(s) ....

(iv) ...

(v) ....

3.4.5 Proofs

A proof of a result should consist of *the introductory paragraph of the proof* (see Section 3.4.5 below), of a finite possibly empty sequence of *argumentation sentences* (see Section 3.4.5 below), and of *the closing sentence* (see Section 3.4.5 below).

The introductory paragraph of the proof Every proof of a result consists of at least two sentences. The beginning of the proof starts

(i) with "Throughout this proof assume w.l.o.g. that ... (1st hypothesis), assume w.l.o.g. that ... (2nd hypothesis), ..., assume w.l.o.g. that ... (k-th hypothesis), let ... (1st set introduction), let ... (2nd set introduction), ..., let ... ($l-1$)-th set introduction), and let ... (l-th set introduction). Note/observe that ..." (where $k, l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$),

(ii) with "Note/observe that ...," or

(iii) with "First, note/observe that ...."

Argumentation sentences In each argumentation sentence we distinguish between

1. a *Type A1 argumentation sentence*,
2. a *Type A2 argumentation sentence*,
3. a *Type B1 argumentation sentence*, and
4. a *Type B2 argumentation sentence*.

An argumentation sentence which employs the previous argumentation sentence is a Type A1 argumentation sentence or a Type A2 argumentation sentence. An argumentation sentence which does not employ the previous argumentation sentence is a Type B1 argumentation sentence or a Type B2 argumentation sentence. Each argumentation sentence may only use the following verbs:

1. note
2. observe
3. imply/implies
4. assure/assures
5. ensure/ensures
6. demonstrate/demonstrates
7. prove/proves
8. establish/establishes

9. show/shows
10. yield/yields
11. hold/holds
12. combine
13. obtain
14. satisfy/satisfies
15. exist/exists
16. be

Besides the above verbs, no other verb may be used in an argumentation sentence. If an argumentation sentence contains the phrase "combine," then this argumentation sentence must contain exactly one of the following phrases:

1. "In the next step we combine … to obtain that … ."
2. "Next, we combine … to obtain that … ."

The phrase "combine" may only appear in a Type B1 argumentation sentence.

Type A1 argumentation sentences A Type A1 argumentation sentence must contain exactly one of the following phrases:

1. "therefore"
2. "Therefore, we obtain that … "
3. "hence"
4. "Hence, we obtain that … "
5. "this"
6. "This … "

Each Type A1 argumentation sentence must use exactly one of the following structures:

1. Hence, we obtain that … (followed by possible quantification of a predicate)
2. Therefore, we obtain that … (followed by possible quantification of a predicate)
3. Combining S_1, S_2, …, and S_n (where $n \in \{2, 3, 4, \dots\}$) hence/therefore implies/ensures/assures/demonstrates/proves/establishes/yields/shows (followed by possible quantification of a predicate)

4. Combining S_1, S_2, $\ldots$, and S_n (where $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$) with T_1, T_2, $\ldots$, and T_k (where $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$) hence/therefore implies/ensures/assures/demonstrates/proves/establishes/yields/shows (followed by possible quantification of a predicate)

5. Combining this, S_1, S_2, $\ldots$, and S_n (where $n \in \mathbb{N}$) implies/ensures/assures/demonstrates/proves/establishes/yields/shows (followed by possible quantification of a predicate)

6. Combining this, S_1, S_2, $\ldots$, and S_n (where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) with T_1, T_2, $\ldots$, and T_k (where $k \in \mathbb{N}$) implies/ensures/assures/demonstrates/proves/establishes/yields/shows (followed by possible quantification of a predicate)

7. This implies/ensures/assures/demonstrates/proves/establishes/yields/shows that (followed by possible quantification of a predicate)

8. This, S_1, S_2, $\ldots$, and S_n (where $n \in \mathbb{N}$) imply/ensure/assure/demonstrate/prove/establish/yield/show (followed by possible quantification of a predicate)

9. S_1, S_2, $\ldots$, and S_n (where $n \in \{2, 3, 4, \ldots\}$) hence/therefore imply/ensure/assure/demonstrate/prove/establish/yield/show (followed by possible quantification of a predicate) (For example: Lemma 1 and Theorem 3 hence prove that for all $x \in \mathbb{R}$ it holds that $f(x) = y$.)

10. S_1 hence/therefore implies/ensures/assures/demonstrates/proves/establishes/yields/shows (followed by possible quantification of a predicate) (For example: Lemma 1 hence proves that for all $x \in \mathbb{R}$ it holds that $f(x) = y$.)

The phrase "This" may only appear in a Type A1 argumentation sentence. The phrase "this" may only appear in a Type A1 argumentation sentence.

Type A2 argumentation sentences A Type A2 argumentation sentence must follow the following structure: "Observe/Note that … imply/implies/ensure/ensures/demonstrate/demonstrates/show/shows/prove/proves/yield/yields … ."

Type B1 argumentation sentences A Type B1 argumentation sentence must contain exactly one of the following phrases:

1. "Moreover, note that …"

2. "Moreover, observe that …"

3. "Furthermore, note that …"

4. "Furthermore, observe that …"

5. "In addition, note that …"

6. "In addition, observe that …"

7. "Next, note that . . . "
8. "Next, observe that . . . "
9. "In the next step, we note that . . . "
10. "In the next step, we observe that . . . "
11. "In the next step, we combine . . . to obtain that . . . "
12. "Next, we combine . . . to obtain that . . . "

Type B2 argumentation sentences A Type B2 argumentation sentence must follow the following structure: "In the next step/Next, let . . . which satisfy/satisfies . . . ."

The closing sentence of the proof The last sentence of the proof reads

1. as "The proof of . . . is thus complete." or
2. as "This completes the proof of . . . ."

4 An illustrative example

We now provide an illustrative example of how these writing styles may be used in a scientific article/thesis. Note that in an actual article/thesis, we would utilize `\section{...}` rather than `\subsection{...}`.

4.1 Introduction

Here should be an introduction.

4.1.1 Notation

Throughout this article/thesis we use the following notation. For sets A and B we denote by $\mathbb{M}(A, B)$ the set of all functions from A to B. Moreover, for measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A}, \mathcal{B})$ the set of all $\mathcal{A}/\mathcal{B}$-measurable spaces.

4.1.2 Setting

Throughout this article/thesis the following setting is frequently used. Let $T \in (0, \infty)$, $m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, and let $W\colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion.

4.2 Results

Proposition 4.1 (Young's inequality)**.** Let $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$, $p_1, \ldots, p_n \in (1, \infty)$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$. Then it holds that

$$a_1 \cdot a_2 \cdot \ldots \cdot a_n \leq \frac{|a_1|^{p_1}}{p_1} + \ldots + \frac{|a_n|^{p_n}}{p_n}. \tag{4.1}$$

Proof of Proposition 4.1. Throughout this proof without loss of generality assume that $a_1 \cdot a_2 \cdot \ldots \cdot a_n \neq 0$ and let $x_1, \ldots, x_n \in (0, \infty)$ satisfy $x_1 = \ln(|a_1|^{p_1})$, $x_2 = \ln(|a_2|^{p_2})$, …, $x_n = \ln(|a_n|^{p_n})$. This ensures that for all $k \in \{1, 2, \ldots, n\}$ it holds that $a_k \neq 0$. Next, note that the convexity of the function $(0, \infty) \ni x \mapsto e^x \in (0, \infty)$ and the assumption that $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$ ensure that

$$\begin{aligned} a_1 \cdot a_2 \cdot \ldots \cdot a_n &\leq |a_1 \cdot a_2 \cdot \ldots \cdot a_n| \\ &= \exp\left(\frac{\ln(|a_1|^{p_1})}{p_1}\right) \cdot \exp\left(\frac{\ln(|a_2|^{p_2})}{p_2}\right) \cdot \ldots \cdot \exp\left(\frac{\ln(|a_n|^{p_n})}{p_n}\right) \\ &= \exp\left(\frac{x_1}{p_1}\right) \cdot \exp\left(\frac{x_2}{p_2}\right) \cdot \ldots \cdot \exp\left(\frac{x_n}{p_n}\right) = \exp\left(\frac{x_1}{p_1} + \frac{x_2}{p_2} + \ldots + \frac{x_n}{p_n}\right) \\ &\leq \frac{\exp(x_1)}{p_1} + \frac{\exp(x_1)}{p_1} + \ldots + \frac{\exp(x_1)}{p_1} = \frac{|a_1|^{p_1}}{p_1} + \ldots + \frac{|a_n|^{p_n}}{p_n}. \end{aligned} \tag{4.2}$$

The proof of Proposition 4.1 is thus complete. □

More results on stochastic differential equations can, e.g., be found in Kloeden and Platen [1].

Proposition 4.2 (Lyapunov-type functions)**.** Let $c, T \in [0,\infty)$, $d, m \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be an open set, let $\mu \in \mathcal{M}(\mathcal{B}(D), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(D), \mathcal{B}(\mathbb{R}^{d\times m}))$, $V \in C^2(D, [0,\infty)$ satisfy that for all $x \in D$ it holds that

$$V'(x)\mu(x) + \sum_{k=1}^{m} V''(x)(\sigma_k(x), \sigma_k(x)) \leq c\, V(x), \tag{4.3}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, and let $X\colon [0,T]\times\Omega \to D$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths satisfying that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = X_0 + \int_0^t \mu(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s. \tag{4.4}$$

Then for all $t \in [0,T]$ it holds that

$$\mathbb{E}\big[V(X_t)\big] \leq e^{ct}\,\mathbb{E}\big[V(X_0)\big]. \tag{4.5}$$

Proposition 4.3 (Lyapunov-type functions revisited)**.** Assume the setting outlined in Section 4.1.2, let $c \in [0,\infty)$, $d \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be an open set, let $\mu \in \mathcal{M}(\mathcal{B}(D), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(D), \mathcal{B}(\mathbb{R}^{d\times m}))$, $V \in C^2(D, [0,\infty))$ satisfy that for all $x \in D$ it holds that

$$V'(x)\mu(x) + \sum_{k=1}^{m} V''(x)(\sigma_k(x), \sigma_k(x)) \leq c\, V(x), \tag{4.6}$$

and let $X\colon [0,T]\times\Omega \to D$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths satisfying that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = X_0 + \int_0^t \mu(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s. \tag{4.7}$$

Then it holds for all $t \in [0,T]$ that

$$\mathbb{E}\big[V(X_t)\big] \leq e^{ct}\,\mathbb{E}\big[V(X_0)\big]. \tag{4.8}$$

Theorem 4.4. Assume the setting outlined in Section 4.1.2 and let $\alpha \in \mathbb{R}$. Then

$$\lim_{N\to\infty}\big(N^\alpha\,\mathbb{E}\big[|W_{T/N}|\big]\big) = \begin{cases} \infty & : \alpha \geq 1/2 \\ \mathbb{E}\big[|W_T|\big] & : \alpha = 1/2 \\ 0 & : \alpha < 1/2 \end{cases}. \tag{4.9}$$

References

[1] Kloeden, P. E., and Platen, E. *Numerical solution of stochastic differential equations*, vol. 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992.