Strong L^p-error analysis of nonlinear Monte Carlo approximations for high-dimensional semilinear partial differential equations

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Abstract

Full-history recursive multilevel Picard (MLP) approximation schemes have been shown to overcome the curse of dimensionality in the numerical approximation of high-dimensional semilinear partial differential equations (PDEs) with general time horizons and Lipschitz continuous nonlinearities. However, each of the error analyses for MLP approximation schemes in the existing literature studies the L^2-root-mean-square distance between the exact solution of the PDE under consideration and the considered MLP approximation and none of the error analyses in the existing literature provides an upper bound for the more general L^p-distance between the exact solution of the PDE under consideration and the considered MLP approximation. It is the key contribution of this article to extend the L^2-error analysis for MLP approximation schemes in the literature to a more general L^p-error analysis with $p \in (0, \infty)$. In particular, the main result of this article proves that the proposed MLP approximation scheme indeed overcomes the curse of dimensionality in the numerical approximation of high-dimensional semilinear PDEs with the approximation error measured in the L^p-sense with $p \in (0, \infty)$.

Contents

1 Introduction

It is one of the most challenging topics in computational mathematics to design and analyze algorithms for the approximative solution of high-dimensional partial differential equations (PDEs) and there are several promising approaches to this topic in the scientific literature.

We refer, for instance, to Darbon & Osher [12] for approximation methods for certain high-dimensional first-order Hamilton–Jacobi–Bellman PDEs. We refer, for instance, to [10, 37, 38, 39] and the references mentioned therein for approximation methods for PDEs based on density estimations and particle systems. We refer, for instance, to [8, 9, 11, 20, 36] and the references mentioned therein for approximation methods based on Picard iterations and suitable projections on function spaces. We refer, for instance, to [10, 24, 25, 26, 45, 48] and the references mentioned therein for approximation methods for semilinear parabolic PDEs based on branching diffusion approximations. We refer, for instance, to [46, 47] for approximation methods for semilinear parabolic PDEs based on standard Monte Carlo approximations for nested conditional expectations. We refer, for instance, to [5, 13, 14, 17, 18, 22, 34, 35, 42, 44] and the references therein for deep learning-based approximation methods for high-dimensional PDEs. We refer, for instance, to [15, 16, 30] for full-history recursive multilevel Picard approximation methods for semilinear parabolic PDEs (in the following we abbreviate *full-history recursive multilevel Picard* by MLP).

Standard numerical approximation methods for high-dimensional nonlinear PDEs in the scientific literature suffer from the *curse of dimensionality* (cf., e.g., Bellman [7], Novak & Wozniakowski [41, Chapter 1], and Novak & Ritter [40]) in the sense that the number of computational operations required to approximately compute the PDE solution by means of the considered numerical approximation method grows at least exponentially in the reciprocal $1/\varepsilon$ of the prescribed approximation accuracy $\varepsilon \in (0, \infty)$ or the PDE dimension $d \in \mathbb{N} = \{1, 2, 3, \dots\}$. As of today, to the best of our knowledge, MLP approximation schemes are the only approximation schemes for high-dimensional PDEs in the scientific literature for which it has been proven that they overcome the curse of dimensionality in the numerical approximation of semilinear heat PDEs with general time horizons and Lipschitz continuous nonlinearities in the sense that the number of computational operations required to approximately compute the PDE solution using MLP approximation schemes grows at most polynomially in both the reciprocal $1/\varepsilon$ of the prescribed approximation accuracy $\varepsilon \in (0, \infty)$ and the dimension $d \in \mathbb{N}$ of the PDE; cf. Hutzenthaler et al. [30]. This type of scaling of computational complexity is also referred to as *polynomial tractability* in the scientific literature (cf., e.g., Novak & Wozniakowski [41, Definition 4.44]).

The complexity analysis in [30] has been extended to more general MLP approximation schemes and more general classes of nonlinear PDEs. More specifically, we refer to [28, 31] for complexity analyses for MLP approximation schemes for parabolic semilinear PDEs involving more general second-order differential operators than just the Laplacian, we refer to Beck et al. [3] for complexity analyses for MLP approximation schemes for parabolic semilinear PDEs with possibly non-Lipschitz continuous nonlinearities such as Allen-Cahn equations, we refer to Beck et al. [2] for complexity analyses for MLP approximation schemes for elliptic semilinear PDEs with Lipschitz continuous nonlinearities, we refer to [27, 32] for complexity analyses for MLP approximation schemes for parabolic semilinear PDEs with gradient-dependent nonlinearities, and we refer to Giles et al. [19] for complexity analyses for a general class of MLP approximation schemes for semilinear heat PDEs. We also refer to [6, 15] for numerical simulations for MLP approximation schemes. Each of the error analyses for MLP approximation schemes in the above-mentioned articles studies the L^2-root-mean-square distance between the exact solution of the PDE under consideration and the considered MLP approximation and none of the error analyses in the above-mentioned articles provides an upper bound for the more general L^p-distance where $p \in (0, \infty)$ between the exact solution of the PDE under consideration and the considered MLP approximation.

It is precisely the subject of this article to extend the L^2-error analyses for MLP approximation schemes in [30] to a more general L^p-error analysis with $p \in (0, \infty)$ and, thereby, also introduce a slightly different variation of the previously studied MLP approximation schemes; see (1.2) below.

It turns out that it is not straightforward to extend the L^2-error analysis for MLP approximation schemes from the literature to a more general L^p-error analysis with $p \in [2, \infty)$ (cf., e.g., Rio [43, Theorem 2.1]). A central difficulty is related to the issue that in our L^p-error analysis the growth of the number of samples used to approximate expectations via Monte Carlo averages must be more carefully chosen; see (1.9) and (1.10) below for details.

To better illustrate the findings of this work, we present in the following result, Theorem 1.1 below, a special case of Theorem 4.6, the main result of this paper. Below Theorem 1.1 we add some explanatory comments regarding the statement of Theorem 1.1 and the mathematical objects appearing in Theorem 1.1 and we also present a brief sketch of our proof of Theorem 1.1.

Theorem 1.1. *Let $T, \kappa, \delta, p \in (0,\infty)$, $\Theta = \bigcup_{n\in\mathbb{N}} \mathbb{Z}^n$, let $f\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, let $u_d \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, $d\in\mathbb{N}$, satisfy for all $d\in\mathbb{N}$, $t\in[0,T]$, $x=(x_1,x_2,\dots,x_d)\in\mathbb{R}^d$ that $|u_d(t,x)| \le \kappa d^\kappa (1+\sum_{k=1}^d |x_k|)^\kappa$ and*

$$(\tfrac{\partial}{\partial t} u_d)(t,x) = (\Delta_x u_d)(t,x) + f(u_d(t,x)), \tag{1.1}$$

let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\mathfrak{u}^\theta\colon \Omega\to[0,1]$, $\theta\in\Theta$, be i.i.d. random variables, assume for all $r\in(0,1)$ that $\mathbb{P}(\mathfrak{u}^0 \le r) = r$, let $W^{d,\theta}\colon [0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $\theta\in\Theta$, be independent standard Brownian motions, assume that $(\mathfrak{u}^\theta)_{\theta\in\Theta}$ and $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ are independent, let $\phi\colon\mathbb{N}\to\mathbb{N}$ and $U^{d,\theta}_{n,m}\colon[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $d,n,m\in\mathbb{Z}$, $\theta\in\Theta$, satisfy for all $n\in\mathbb{N}_0$, $d,m\in\mathbb{N}$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $\phi(m)=\max\{k\in\mathbb{N}\colon k\le\exp(|\ln(m)|^{1/2})\}$ and

$$\begin{aligned} U^{d,\theta}_{n,m}(t,x) &= \sum_{i=0}^{n-1} \tfrac{t}{(\phi(m))^{n-i}} \Bigg[\sum_{k=1}^{(\phi(m))^{n-i}} \Big[f\big(U^{d,(\theta,i,k)}_{i,m}(t\mathfrak{u}^{(\theta,i,k)}, x+\sqrt{2}\,W^{d,(\theta,i,k)}_{t-t\mathfrak{u}^{(\theta,i,k)}})\big) \\ &- \mathbb{1}_{\mathbb{N}}(i)\, f\big(U^{d,(\theta,-i,k)}_{i-1,m}(t\mathfrak{u}^{(\theta,i,k)}, x+\sqrt{2}\,W^{d,(\theta,i,k)}_{t-t\mathfrak{u}^{(\theta,i,k)}})\big)\Big]\Bigg] + \tfrac{\mathbb{1}_{\mathbb{N}}(n)}{(\phi(m))^n} \Bigg[\sum_{k=1}^{(\phi(m))^n} u_d\big(0, x+\sqrt{2}\,W^{d,(\theta,0,-k)}_t\big)\Bigg], \end{aligned} \tag{1.2}$$

and for every $d,n,m\in\mathbb{N}$ let $\mathfrak{C}_{d,n,m}\in\mathbb{N}$ be the number of function evaluations of f and $u_d(0,\cdot)$ and the number of realizations of scalar random variables which are used to compute one realization of $U^{d,0}_{n,m}(T,0)\colon\Omega\to\mathbb{R}$ (see (4.28) for a precise definition). Then there exist $c\in\mathbb{R}$ and $\mathfrak{n}\colon\mathbb{N}\times(0,1]\to\mathbb{N}$ such that for all $d\in\mathbb{N}$, $\varepsilon\in(0,1]$ it holds that

$$\big(\mathbb{E}\big[|u_d(T,0) - U^{d,0}_{\mathfrak{n}(d,\varepsilon),\mathfrak{n}(d,\varepsilon)}(T,0)|^p\big]\big)^{1/p} \le \varepsilon \qquad \text{and} \qquad \mathfrak{C}_{d,\mathfrak{n}(d,\varepsilon),\mathfrak{n}(d,\varepsilon)} \le c d^c \varepsilon^{-(2+\delta)}. \tag{1.3}$$

Theorem 1.1 is an immediate consequence of Theorem 4.6 in Section 4 below. Theorem 4.6, which is the main result of this article, in turn, follows from Proposition 4.4 (see Section 4 below for details). In the following we provide some explanatory comments concerning the mathematical objects appearing in Theorem 1.1 above.

In Theorem 1.1 we intend to approximate the solutions of the PDEs in (1.1). The strictly positive real number $T\in(0,\infty)$ in Theorem 1.1 describes the time horizon of the PDEs in (1.1), the Lipschitz continuous function $f\colon\mathbb{R}\to\mathbb{R}$ specifies the nonlinearity of the PDEs in (1.1), and the functions $u_d\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $d\in\mathbb{N}$, are the solutions of the PDEs in (1.1).

The strictly positive real number $\kappa\in(0,\infty)$ in Theorem 1.1 is employed to formulate a regularity condition for the solutions $u_d\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $d\in\mathbb{N}$, of the PDEs in (1.1) which we impose in Theorem 1.1. More formally, in Theorem 1.1 we assume that the solution functions $u_d\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $d\in\mathbb{N}$, of the PDEs in (1.1) satisfy the regularity condition that for all $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$|u_d(t,x)| \le \kappa d^\kappa \big(1+\textstyle\sum_{k=1}^d |x_k|\big)^\kappa. \tag{1.4}$$

This condition ensures that the solution functions $u_d\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $d\in\mathbb{N}$, of the PDEs in (1.1) are at most polynomially growing both in the spatial variable $x\in\mathbb{R}^d$ and in the PDE dimension $d\in\mathbb{N}$. Observe that the condition in (1.4) also ensures that solutions of the PDEs in (1.1) with the fixed initial value functions $\mathbb{R}^d\ni x\mapsto u_d(0,x)\in\mathbb{R}$, $d\in\mathbb{N}$, are unique.

In (1.2) we recursively specify the proposed MLP approximations which we employ in Theorem 1.1 to approximate the solutions of the PDEs in (1.1). The proposed MLP approximation method is a random approximation algorithm which is defined on an artificial probability space. The probability space $(\Omega,\mathcal{F},\mathbb{P})$ in Theorem 1.1 is this artificial probability space on which we defined the proposed MLP approximations.

To formulate the proposed MLP approximations, we need, roughly speaking, sufficiently many independent random variables as random input sources and to formulate these sufficiently many independent random variables, we need, roughly speaking, a sufficiently large index set over which the sufficiently many independent random variables are defined. The set $\Theta = \bigcup_{n\in\mathbb{N}}\mathbb{Z}^n$ in Theorem 1.1 is precisely this sufficiently large index set which allows us to introduce sufficiently many independent random variables over this index set and the i.i.d. random variables $\mathfrak{u}^\theta\colon \Omega \to [0,1]$, $\theta\in\Theta$, and the independent standard Brownian motions $W^{d,\theta}\colon [0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $\theta\in\Theta$, are the sufficiently many independent random variables which we use to specify the MLP approximations in (1.2). Observe that the assumption that for all $r\in(0,1)$ it holds that $\mathbb{P}(\mathfrak{u}^0\le r)=r$ in Theorem 1.1 ensures that the random variables $\mathfrak{u}^\theta\colon\Omega\to[0,1]$, $\theta\in\Theta$, are on $[0,1]$ continuous uniformly distributed random variables.

The MLP approximations specified in (1.2) differ from previously introduced MLP approximations, roughly speaking, in the sense that a smaller number of Monte Carlo samples is employed. In Theorem 1.1 this smaller number of Monte Carlo samples is formulated through the function $\phi\colon\mathbb{N}\to\mathbb{N}$ which increases quite slowly to infinity. More formally, Lemma 4.5 in Subsection 4.4 below proves that for every $\varepsilon\in(0,\infty)$ there exists $c\in\mathbb{R}$ such that for all $x\in[1,\infty)$ it holds that $\lim_{y\to\infty}\phi(y)=\infty$ and $\phi(x)\le cx^\varepsilon$. This slow increase to infinity is an important argument in our L^p-error analysis for the proposed MLP approximations (see (1.9), (1.10), and Subsection 4.4 below for further details).

The natural numbers $\mathfrak{C}_{d,n,m}\in\mathbb{N}$, $d,m,n\in\mathbb{N}$, in Theorem 1.1 measure the computational cost of the proposed MLP approximations. More specifically, for every $d,m,n\in\mathbb{N}$ we have that $\mathfrak{C}_{d,n,m}$ is the sum of the number of function evaluations of the nonlinearity $f\colon\mathbb{R}\to\mathbb{R}$, of the number of function evaluations of the initial value function $\mathbb{R}^d\ni x\mapsto u_d(0,x)\in\mathbb{R}$, and of the number of one-dimensional random variables which are used to compute one realization of the MLP approximation $U^{d,0}_{n,m}(T,0)\colon\Omega\to\mathbb{R}$. We also refer to (4.28) in Proposition 4.4 in Section 4 below for the precise specification of the natural numbers $\mathfrak{C}_{d,n,m}\in\mathbb{N}$, $d,m,n\in\mathbb{N}$.

Theorem 1.1 reveals that the MLP approximations in (1.2) approximate the values $u_d(T,0)\in\mathbb{R}$, $d\in\mathbb{N}$, of the solution functions $u_d\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $d\in\mathbb{N}$, at the terminal time $t=T$ and at the space point $x=0\in\mathbb{R}^d$ with a computational effort which grows at most polynomially in the PDE dimension $d\in\mathbb{N}$ and up to an arbitrarily small polynomial order at most quadratically in the reciprocal of the prescribed approximation accuracy $\varepsilon>0$. This arbitrarily small polynomial order is described through the real number $\delta\in(0,\infty)$ in Theorem 1.1.

Due to the fact that the MLP approximations proposed in (1.2) differ slightly from the MLP approximations which have been previously employed in L^2-error analyses in the scientific literature, we now briefly sketch the main ideas in the proof of Theorem 1.1. The first step in our sketch of the proof of Theorem 1.1 is to reformulate the PDEs under consideration as stochastic fixed-point equations. Specifically, in the context of (1.1) we have that the Feynman-Kac formula proves that the solution functions $u_d\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $d\in\mathbb{N}$, of the PDEs in (1.1) are the unique at most polynomially growing functions which satisfy for all $d\in\mathbb{N}$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that

$$u_d(t,x)=\mathbb{E}\big[u_d(0,x+\sqrt{2}W^{d,\theta}_t)\big]+\int_0^t\mathbb{E}\big[f(u_d(s,x+\sqrt{2}W^{d,\theta}_{t-s}))\big]\,ds. \tag{1.5}$$

In the next step we note that (1.2), the assumption that $W^{d,\theta}\colon[0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $\theta\in\Theta$, are independent standard Brownian motions, and the assumption that $\mathfrak{u}^\theta\colon\Omega\to[0,1]$, $\theta\in\Theta$, are i.i.d. random variables assure that for all $n\in\mathbb{N}_0$, $d,m\in\mathbb{N}$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$\begin{aligned}
&\mathbb{E}\big[U^{d,\theta}_{n,m}(t,x)\big]-\mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[u_d\big(0,x+\sqrt{2}W^{d,\theta}_t\big)\big]\\
&=t\left[\sum_{i=0}^{n-1}\mathbb{E}\Big[f\big(U^{d,(\theta,i)}_{i,m}(t\mathfrak{u}^\theta,x+\sqrt{2}W^{d,\theta}_{t-t\mathfrak{u}^\theta})\big)-\mathbb{1}_{\mathbb{N}}(i)f\big(U^{d,(\theta,-i)}_{i-1,m}(t\mathfrak{u}^\theta,x+\sqrt{2}W^{d,\theta}_{t-t\mathfrak{u}^\theta})\big)\Big]\right]\\
&=t\left[\sum_{i=0}^{n-1}\mathbb{E}\Big[f\big(U^{d,\theta}_{i,m}(t\mathfrak{u}^\theta,x+\sqrt{2}W^{d,\theta}_{t-t\mathfrak{u}^\theta})\big)-\mathbb{1}_{\mathbb{N}}(i)f\big(U^{d,\theta}_{i-1,m}(t\mathfrak{u}^\theta,x+\sqrt{2}W^{d,\theta}_{t-t\mathfrak{u}^\theta})\big)\Big]\right]
\end{aligned} \tag{1.6}$$

(cf. Lemmas 3.3 and 3.5 and (3.24) in the proof of Lemma 3.5 for the details). In addition, we observe that the assumption that $W^{d,\theta}\colon[0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $\theta\in\Theta$, are independent standard

Brownian motions, the assumption that $\mathfrak{u}^\theta\colon \Omega \to [0,1]$, $\theta \in \Theta$, are i.i.d. random variables, and a telescoping sum argument demonstrate that for all $n \in \mathbb{N}_0$, $d, m \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$t\left[\sum_{i=0}^{n-1} \mathbb{E}\Big[f\big(U_{i,m}^{d,\theta}(t\mathfrak{u}^\theta, x + \sqrt{2}W_{t-t\mathfrak{u}^\theta}^{d,\theta})\big) - \mathbb{1}_{\mathbb{N}}(i) f\big(U_{i-1,m}^{d,\theta}(t\mathfrak{u}^\theta, x + \sqrt{2}W_{t-t\mathfrak{u}^\theta}^{d,\theta})\big)\Big]\right] \tag{1.7}$$
$$= \mathbb{1}_{\mathbb{N}}(n)\, t\, \mathbb{E}\Big[f\big(U_{n-1,m}^{d,\theta}(t\mathfrak{u}^\theta, x + \sqrt{2}W_{t-t\mathfrak{u}^\theta}^{d,\theta})\big)\Big] = \mathbb{1}_{\mathbb{N}}(n)\left[\int_0^t \mathbb{E}\Big[f\big(U_{n-1,m}^{d,\theta}(s, x + \sqrt{2}W_{t-s}^{d,\theta})\big)\Big]\, ds\right]$$

(cf. Lemmas 3.4 and 3.5). Combining (1.5), (1.6), and (1.7) indicates that for all $n \in \mathbb{N}_0$, $d, m \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ we have that

$$\begin{aligned}
\mathbb{E}\big[U_{n,m}^{d,\theta}(t,x)\big] &= \mathbb{1}_{\mathbb{N}}(n)\bigg(\mathbb{E}\big[u_d\big(0, x + \sqrt{2}W_t^{d,\theta}\big)\big] + \int_0^t \mathbb{E}\Big[f\big(U_{n-1,m}^{d,\theta}(s, x + \sqrt{2}W_{t-s}^{d,\theta})\big)\Big]\, ds\bigg) \\
&\approx \mathbb{1}_{\mathbb{N}}(n)\bigg(\mathbb{E}\big[u_d\big(0, x + \sqrt{2}W_t^{d,\theta}\big)\big] + \int_0^t \mathbb{E}\Big[f\big(u_d(s, x + \sqrt{2}W_{t-s}^{d,\theta})\big)\Big]\, ds\bigg) \\
&= \mathbb{1}_{\mathbb{N}}(n)\, u_d(t,x)
\end{aligned} \tag{1.8}$$

(cf. Lemmas 3.5 and 3.14). Observe that (1.8) suggests that the proposed MLP approximations $U_{n,m}^{d,\theta}\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $n \in \mathbb{N}_0$, $d, m \in \mathbb{N}$, $\theta \in \Theta$, behave in expectation like Picard iterations for the stochastic fixed-point equations in (1.5). The final step in our sketch of the proof of Theorem 1.1 is to employ a Monte Carlo approach to approximate the expectations in (1.6). This final step is where the MLP approximations proposed in (1.2) differ from the MLP approximations which have been previously employed in L^2-error analyses in the scientific literature. Specifically, the MLP approximations proposed in (1.2) use the fact that for all $n \in \mathbb{N}_0$, $i \in \{0, 1, \dots, n-1\}$, $d \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ we have that

$$\tfrac{1}{(\phi(m))^{n-i}} \sum_{k=1}^{(\phi(m))^{n-i}} \Big[f\big(U_{i,m}^{d,(\theta,i,k)}(t\mathfrak{u}^{(\theta,i,k)}, x + \sqrt{2}W_{t-t\mathfrak{u}^{(\theta,i,k)}}^{d,(\theta,i,k)})\big) - \mathbb{1}_{\mathbb{N}}(i) f\big(U_{i-1,m}^{d,(\theta,-i,k)}(t\mathfrak{u}^{(\theta,i,k)}, x + \sqrt{2}W_{t-t\mathfrak{u}^{(\theta,i,k)}}^{d,(\theta,i,k)})\big)\Big] \tag{1.9}$$

is a Monte Carlo approximation of

$$\mathbb{E}\Big[f\big(U_{i,m}^{d,(\theta,i)}(t\mathfrak{u}^\theta, x + \sqrt{2}W_{t-t\mathfrak{u}^\theta}^{d,\theta})\big) - \mathbb{1}_{\mathbb{N}}(i) f\big(U_{i-1,m}^{d,(\theta,-i)}(t\mathfrak{u}^\theta, x + \sqrt{2}W_{t-t\mathfrak{u}^\theta}^{d,\theta})\big)\Big] \tag{1.10}$$

employing $(\phi(m))^{n-i} \in \mathbb{N}$ samples. The function $\phi\colon \mathbb{N} \to \mathbb{N}$ thus determines the number of samples used in the Monte Carlo approximations in the MLP approximations proposed in (1.2).

In our L^p-error analysis the specific choice of ϕ is a subtle issue and, in particular, in our L^p-error analysis there is some fine-tuning needed in the choice of the function ϕ. On the one hand, the function ϕ must be chosen large enough so that the error due to approximating expectations via Monte Carlo averages is small enough. On the other hand, in our recursive Gronwall-type L^p-error analysis in Lemma 3.13 in Subsection 3.6 and Lemma 3.14 in Subsection 3.7 the exponential term $\exp(m^{p/2}/p)$ arises in the upper bounds (see (3.72) in Lemma 3.13, (3.75) in Lemma 3.14, and (4.38) in the proof of Proposition 4.4) where $m \in \mathbb{N}$ will be replaced by $\phi(m)$. To control this term, our L^p-error analysis employs the assumption that $(\phi(m)^{p/2}/m)_{m\in\mathbb{N}}$ is a bounded sequence. More specifically, if $p \in (0,2]$, then ϕ may be chosen to be the identity, but if $p \in (2,\infty)$, then ϕ must grow much slower and the choice $\forall\, m \in \mathbb{N}\colon \phi(m) = \max\{k \in \mathbb{N}\colon k \leq \exp(|\ln(m)|^{1/2})\}$ is a suitable p-independent choice.

The remainder of this article is structured as follows. In Section 2 we establish regularity properties for solutions of stochastic fixed-point equations. Afterwards, in Section 3 we introduce MLP approximations for the stochastic fixed-point equations from Section 2, we study their measurability and integrability properties, and we establish L^p-error bounds between the exact solutions of the stochastic fixed-point equations and the MLP approximations. Finally, in Section 4 we establish some elementary estimates for full-history recursions and combine these estimates with the regularity properties for solutions of stochastic fixed-point equations, which we established in Section 2, and the L^p-error analysis for MLP approximations for stochastic fixed-point equations, which we established in Section 3, to obtain a computational complexity analysis for MLP approximations for semilinear partial differential equations.

2 Stochastic fixed-point equations

In this section we establish in Corollary 2.5 below appropriate regularity results for solutions of stochastic fixed-point equations with polynomially growing solutions. In Corollary 2.5 we assume, among other things, that the nonlinearity $f\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ and the terminal condition $g\colon\mathbb{R}^d\to\mathbb{R}$ of the stochastic fixed-point equation in (2.18) satisfy the polynomial growth bound that there exist $\mathfrak{L},p\in[0,\infty)$ such that for all $t\in[0,T]$, $x=(x_1,x_2,\dots,x_d)\in\mathbb{R}^d$ it holds that

$$\max\{|f(t,x,0)|,|g(x)|\}\leq\mathfrak{L}\big(1+[\textstyle\sum_{k=1}^d|x_k|^2]^{p/2}\big) \tag{2.1}$$

(see above (2.18) in Corollary 2.5). Observe that in the case $x=0\in\mathbb{R}^d$, $p=0$ we have that (2.1) reduces to the condition that for all $t\in[0,T]$ it holds that $\max\{|f(t,0,0)|,|g(0)|\}\leq\mathfrak{L}(1+[\sum_{k=1}^d|0|^2]^{p/2})=\mathfrak{L}(1+0^0)=2\mathfrak{L}$.

Our proof of Corollary 2.5 uses the regularity result for stochastic fixed-point equations with Lipschitz continuous nonlinearities in Lemma 2.3 below. Similar regularity results for stochastic fixed-point equations can, e.g., be found in Hutzenthaler et al. [29, Lemma 2.2]. Our proof of Lemma 2.3 uses the well-known backward formulation of the Gronwall inequality in Corollary 2.2 below. In our proof of Corollary 2.2 we use the well-known forward formulation of the Gronwall inequality in Lemma 2.1. Lemma 2.1 is a direct consequence of, e.g., the generalized Gronwall inequality in Henry [23, Lemma 7.1.1].

2.1 Gronwall-type inequalities

Lemma 2.1. *Let $T,\gamma\in[0,\infty)$, let $\beta\colon[0,T]\to[0,\infty)$ be a function, let $\alpha\colon[0,T]\to[0,\infty]$ be measurable, and assume for all $t\in[0,T]$ that $\int_0^t\alpha(s)\,ds<\infty$ and*

$$\alpha(t)\leq\beta(t)+\gamma\int_0^t\alpha(s)\,ds. \tag{2.2}$$

Then it holds for all $t\in[0,T]$ that $\alpha(t)\leq[\sup_{s\in[0,t]}\beta(s)]\exp(\gamma t)$.

Corollary 2.2. *Let $T,\gamma\in[0,\infty)$, let $\beta\colon[0,T]\to[0,\infty)$ be non-increasing, let $\alpha\colon[0,T]\to[0,\infty]$ be measurable, and assume for all $t\in[0,T]$ that $\int_t^T\alpha(s)\,ds<\infty$ and*

$$\alpha(t)\leq\beta(t)+\gamma\int_t^T\alpha(s)\,ds. \tag{2.3}$$

Then it holds for all $t\in[0,T]$ that $\alpha(t)\leq\beta(t)\exp(\gamma(T-t))<\infty$.

Proof of Corollary 2.2. Throughout this proof let $\Lambda\colon[0,T]\to[0,\infty]$ and $\mathfrak{a}\colon[0,T]\to[0,\infty)$ satisfy for all $t\in[0,T]$ that

$$\Lambda(t)=\alpha(T-t)\qquad\text{and}\qquad\mathfrak{a}(t)=\beta(T-t). \tag{2.4}$$

Note that the hypothesis that α is measurable and (2.4) ensure that Λ is measurable. In addition, observe that the hypothesis that for all $t \in [0,T]$ it holds that $\int_t^T \alpha(s)\,ds < \infty$ and (2.4) assure that for all $t \in [0,T]$ it holds that

$$\int_0^t \Lambda(s)\,ds = \int_0^t \alpha(T-s)\,ds = \int_{T-t}^T \alpha(s)\,ds \le \int_0^T \alpha(s)\,ds < \infty. \tag{2.5}$$

Moreover, note that the hypothesis that β is non-increasing and (2.4) guarantee that $\mathfrak{a}$ is non-decreasing. Furthermore, observe that (2.3), (2.4), and (2.5) demonstrate that for all $t \in [0,T]$ it holds that

$$\Lambda(t) = \alpha(T-t) \le \beta(T-t) + \gamma \int_{T-t}^T \alpha(s)\,ds = \mathfrak{a}(t) + \gamma \int_0^t \alpha(T-s)\,ds = \mathfrak{a}(t) + \gamma \int_0^t \Lambda(s)\,ds. \tag{2.6}$$

This, (2.4), the fact that Λ is measurable, (2.5), the fact that $\mathfrak{a}$ is non-decreasing, and Lemma 2.1 (applied with $T \curvearrowleft T$, $\gamma \curvearrowleft \gamma$, $\beta \curvearrowleft \mathfrak{a}$, $\alpha \curvearrowleft \Lambda$ in the notation of Lemma 2.1) prove that for all $t \in [0,T]$ it holds that

$$\Lambda(t) \le \left[\sup_{s\in[0,t]} \mathfrak{a}(s)\right] \exp(\gamma t) = \mathfrak{a}(t)\exp(\gamma t) < \infty. \tag{2.7}$$

Combining this and (2.4) establishes that for all $t \in [0,T]$ it holds that

$$\alpha(t) \le \beta(t)\exp(\gamma(T-t)) < \infty. \tag{2.8}$$

The proof of Corollary 2.2 is thus complete. □

2.2 A priori bounds for solutions of stochastic fixed-point equations

Lemma 2.3. *Let $d \in \mathbb{N}$, $T, L \in [0,\infty)$, $q \in [1,\infty)$, $f \in C([0,T] \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}^d,\mathbb{R})$, $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^d$ be a standard Brownian motion, and assume for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(t,x,v) - f(t,x,w)| \le L|v-w|$, $\mathbb{E}[|g(x+W_{T-t})| + \int_t^T |f(s,x+W_{s-t},u(s,x+W_{s-t}))|\,ds] + \int_t^T (\mathbb{E}[|u(s,x+W_s)|^q])^{1/q}\,ds < \infty$, and*

$$u(t,x) = \mathbb{E}\big[g(x+W_{T-t})\big] + \int_t^T \mathbb{E}\big[f(s,x+W_{s-t},u(s,x+W_{s-t}))\big]\,ds. \tag{2.9}$$

Then it holds for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\Big(\mathbb{E}\Big[|u(t,x+W_t)|^q\Big]\Big)^{1/q} \qquad (2.10)\\ &\le \exp\big(L(T-t)\big)\left[\Big(\mathbb{E}\Big[|g(x+W_T)|^q\Big]\Big)^{1/q} + (T-t)^{(q-1)/q}\bigg(\int_t^T \mathbb{E}\Big[|f(s,x+W_s,0)|^q\Big]\,ds\bigg)^{1/q}\right]. \end{aligned}$$

Proof of Lemma 2.3. Throughout this proof let $\alpha\colon [0,T]\times\mathbb{R}^d \to [0,\infty]$ satisfy for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\alpha(t,x) = \big(\mathbb{E}\big[|g(x+W_T)|^q\big]\big)^{1/q} + (T-t)^{(q-1)/q}\bigg(\int_t^T \mathbb{E}\big[|f(s,x+W_s,0)|^q\big]\,ds\bigg)^{1/q} \tag{2.11}$$

and assume without loss of generality that for all $x \in \mathbb{R}^d$ it holds that $\alpha(0,x) < \infty$. Note that (2.11) ensures that for all $x \in \mathbb{R}^d$ it holds that $[0,T] \ni t \mapsto \alpha(t,x) \in [0,\infty)$ is non-increasing. In addition, observe that (2.9), the triangle inequality, Jensen's inequality, Fubini's theorem, and the fact that W has independent increments assure that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\big(\mathbb{E}\big[|u(t,x+W_t)|^q\big]\big)^{1/q}$$

$$
\begin{aligned}
&= \Big(\mathbb{E}\Big[\Big|\mathbb{E}\big[g(x+W_{T-t}+W_t)\big] + \textstyle\int_t^T \mathbb{E}\big[f(s,x+W_{s-t}+W_t,u(s,x+W_{s-t}+W_t))\big]\,ds\Big|^q\Big]\Big)^{1/q}\\
&= \Big(\mathbb{E}\Big[\Big|\mathbb{E}\big[g(x+W_T)\big] + \textstyle\int_t^T \mathbb{E}\big[f(s,x+W_s,u(s,x+W_s))\big]\,ds\Big|^q\Big]\Big)^{1/q}\\
&\le \Big(\mathbb{E}\Big[\big|\mathbb{E}\big[g(x+W_T)\big]\big|^q\Big]\Big)^{1/q} + \Big(\mathbb{E}\Big[\big|\textstyle\int_t^T \mathbb{E}\big[f(s,x+W_s,u(s,x+W_s))\big]\,ds\big|^q\Big]\Big)^{1/q}\\
&\le \big(\mathbb{E}\big[|g(x+W_T)|^q\big]\big)^{1/q} + \int_t^T \big(\mathbb{E}\big[|f(s,x+W_s,u(s,x+W_s))|^q\big]\big)^{1/q}\,ds.
\end{aligned} \tag{2.12}
$$

Next note that the triangle inequality and the hypothesis that for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $v,w \in \mathbb{R}$ it holds that $|f(t,x,v) - f(t,x,w)| \le L|v-w|$ demonstrate that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$
\begin{aligned}
&\int_t^T \big(\mathbb{E}\big[|f(s,x+W_s,u(s,x+W_s))|^q\big]\big)^{1/q}\,ds \le \int_t^T \big(\mathbb{E}\big[|f(s,x+W_s,0))|^q\big]\big)^{1/q}\,ds\\
&+ \int_t^T \big(\mathbb{E}\big[|f(s,x+W_s,u(s,x+W_s)) - f(s,x+W_s,0)|^q\big]\big)^{1/q}\,ds\\
&\le \int_t^T \big(\mathbb{E}\big[|f(s,x+W_s,0))|^q\big]\big)^{1/q}\,ds + L\int_t^T \big(\mathbb{E}\big[|u(s,x+W_s)|^q\big]\big)^{1/q}\,ds.
\end{aligned} \tag{2.13}
$$

Furthermore, observe that Hölder's inequality shows that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$
\begin{aligned}
\int_t^T \big(\mathbb{E}\big[|f(s,x+W_s,0))|^q\big]\big)^{1/q}\,ds &= \bigg(\bigg[\int_t^T \big(\mathbb{E}\big[|f(s,x+W_s,0))|^q\big]\big)^{1/q}\,ds\bigg]^q\bigg)^{1/q}\\
&\le \bigg((T-t)^{q-1}\int_t^T \mathbb{E}\big[|f(s,x+W_s,0))|^q\big]\,ds\bigg)^{1/q}\\
&= (T-t)^{(q-1)/q}\bigg(\int_t^T \mathbb{E}\big[|f(s,x+W_s,0))|^q\big]\,ds\bigg)^{1/q}.
\end{aligned} \tag{2.14}
$$

Combining this, (2.11), (2.12), and (2.13) guarantees that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$
\big(\mathbb{E}\big[|u(t,x+W_t)|^q\big]\big)^{1/q} \le \alpha(t,x) + L\int_t^T \big(\mathbb{E}\big[|u(s,x+W_s)|^q\big]\big)^{1/q}\,ds. \tag{2.15}
$$

This, (2.11), the fact that for all $x \in \mathbb{R}^d$ it holds that $[0,T] \ni t \mapsto \alpha(t,x) \in [0,\infty)$ is non-increasing, the hypothesis that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $\int_t^T (\mathbb{E}[|u(s,x+W_s)|^q])^{1/q}\,ds < \infty$, and Corollary 2.2 (applied for every $x \in \mathbb{R}^d$ with $T \curvearrowleft T$, $\gamma \curvearrowleft L$, $\beta \curvearrowleft ([0,T] \ni t \mapsto \alpha(t,x) \in [0,\infty))$, $\alpha \curvearrowleft ([0,T] \ni t \mapsto (\mathbb{E}[|u(t,x+W_t)|^q])^{1/q} \in [0,\infty])$ in the notation of Corollary 2.2) establish that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$
\big(\mathbb{E}\big[|u(t,x+W_t)|^q\big]\big)^{1/q} \le \alpha(t,x)\exp\big(L(T-t)\big). \tag{2.16}
$$

The proof of Lemma 2.3 is thus complete. □

Definition 2.4. *We denote by* $\|\cdot\| \colon (\bigcup_{d\in\mathbb{N}}\mathbb{R}^d) \to [0,\infty)$ *the function which satisfies for all* $d \in \mathbb{N}$, $x = (x_1,x_2,\dots,x_d) \in \mathbb{R}^d$ *that* $\|x\| = [\sum_{k=1}^d |x_k|^2]^{1/2}$.

Corollary 2.5. *Let* $d \in \mathbb{N}$, $T, L, \mathfrak{L}, p \in [0,\infty)$, $q \in [1,\infty)$, $f \in C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R}^d,\mathbb{R})$, $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, *let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space, let* $W \colon [0,T]\times\Omega \to \mathbb{R}^d$ *be a standard Brownian motion, and assume for all* $t \in [0,T]$, $x \in \mathbb{R}^d$, $v,w \in \mathbb{R}$ *that* $|f(t,x,v) - f(t,x,w)| \le$

$L|v-w|$, $\max\{|f(t,x,0)|,|g(x)|\} \leq \mathfrak{L}(1+\|x\|^p)$, $\mathbb{E}[|g(x+W_{T-t})| + \int_t^T |f(s,x+W_{s-t},u(s,x+W_{s-t}))|\,ds] < \infty$, and

$$u(t,x) = \mathbb{E}\big[g(x+W_{T-t})\big] + \int_t^T \mathbb{E}\big[f(s,x+W_{s-t},u(s,x+W_{s-t}))\big]\,ds \tag{2.17}$$

(cf. Definition 2.4). Then it holds for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\Big(\mathbb{E}\Big[\big|u(t,x+W_t)\big|^q\Big]\Big)^{1/q} \leq \mathfrak{L}(T+1)\exp(LT)\left[\sup_{s\in[0,T]}\Big(\mathbb{E}\Big[\big(1+\|x+W_s\|^p\big)^q\Big]\Big)^{1/q}\right] < \infty. \tag{2.18}$$

Proof of Corollary 2.5. Throughout this proof let $\mathbb{F}_t \subseteq \mathcal{F}$, $t \in [0,T]$, satisfy for all $t \in [0,T]$ that

$$\mathbb{F}_t = \begin{cases} \bigcap_{s\in(t,T]} \sigma\big(\sigma(W_r\colon r\in[0,s]) \cup \{A \in \mathcal{F}\colon \mathbb{P}(A)=0\}\big) & : t<T \\ \sigma\big(\sigma(W_s\colon s\in[0,T]) \cup \{A \in \mathcal{F}\colon \mathbb{P}(A)=0\}\big) & : t=T \end{cases} \tag{2.19}$$

and let $a \in C([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ and $b \in C([0,T]\times\mathbb{R}^d,\mathbb{R}^{d\times d})$ satisfy for all $t\in[0,T]$, $x,v\in\mathbb{R}^d$ that $a(t,x)=0$ and $b(t,x)v=v$. Note that (2.19) guarantees that $\mathbb{F}_t \subseteq \mathcal{F}$, $t\in[0,T]$, satisfies that

(I) it holds that $\{A\in\mathcal{F}\colon \mathbb{P}(A)=0\} \subseteq \mathbb{F}_0$ and

(II) it holds for all $t \in [0,T]$ that $\mathbb{F}_t = \cap_{s\in(t,T]}\mathbb{F}_s$.

Combining items (I) and (II), (2.19), and, e.g., Hutzenthaler et al. [31, Lemma 2.17] (applied with $m \curvearrowleft d$, $T \curvearrowleft T$, $W \curvearrowleft W$, $\mathbb{H}_t \curvearrowleft \mathbb{F}_t$, $(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]}) \curvearrowleft (\Omega,\mathcal{F},\mathbb{P},(\sigma(W_s\colon s\in[0,t])\cup\{A\in\mathcal{F}\colon \mathbb{P}(A)=0\})_{t\in[0,T]})$ in the notation of [31, Lemma 2.17]) hence assures that $W\colon [0,T]\times\Omega\to\mathbb{R}$ is a standard $(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]})$-Brownian motion. Combining this, the hypothesis that for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $v,w\in\mathbb{R}$ it holds that $|f(t,x,v)-f(t,x,w)| \leq L|v-w|$, the hypothesis that for all $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that $\max\{|f(t,x,0)|,|g(x)|\} \leq \mathfrak{L}(1+\|x\|^p)$, and Beck et al. [4, Corollary 3.9] (applied with $d \curvearrowleft d$, $m \curvearrowleft d$, $T \curvearrowleft T$, $L \curvearrowleft \max\{d^{1/2},L\}$, $\mathfrak{C} \curvearrowleft 0$, $f \curvearrowleft f$, $g \curvearrowleft g$, $\mu \curvearrowleft a$, $\sigma \curvearrowleft b$, $W \curvearrowleft W$, $(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]}) \curvearrowleft (\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]})$ in the notation of [4, Corollary 3.9]) ensures that

$$\sup_{s\in[0,T]}\sup_{y\in\mathbb{R}^d}\left(\frac{|u(s,y)|}{1+\|y\|^p}\right) < \infty. \tag{2.20}$$

This, the fact that for all $r,v,w\in[0,\infty)$ it holds that $(v+w)^r \leq 2^{\max\{r-1,0\}}(v^r+w^r)$, the triangle inequality, and the fact that for all $r\in[0,\infty)$ it holds that $\mathbb{E}[\|W_T\|^r] < \infty$ demonstrate that for all $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$\begin{aligned}
&\int_t^T \big(\mathbb{E}\big[|u(s,x+W_s)|^q\big]\big)^{1/q}\,ds \leq \left[\sup_{s\in[0,T]}\sup_{y\in\mathbb{R}^d}\frac{|u(s,y)|}{1+\|y\|^p}\right]\int_0^T \big(\mathbb{E}\big[(1+\|x+W_s\|^p)^q\big]\big)^{1/q}\,ds \\
&\leq \left[\sup_{s\in[0,T]}\sup_{y\in\mathbb{R}^d}\frac{|u(s,y)|}{1+\|y\|^p}\right]\int_0^T \Big[1+2^{\max\{p-1,0\}}\big(\mathbb{E}\big[(\|x\|^p+\|W_s\|^p)^q\big]\big)^{1/q}\Big]\,ds \\
&\leq T\left[\sup_{s\in[0,T]}\sup_{y\in\mathbb{R}^d}\frac{|u(s,y)|}{1+\|y\|^p}\right]\Big[1+2^{\max\{p-1,0\}}\|x\|^p + 2^{\max\{p-1,0\}}\big(\mathbb{E}\big[\|W_T\|^{pq}\big]\big)^{1/q}\Big] < \infty.
\end{aligned} \tag{2.21}$$

Combining this, the hypothesis that for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $v,w\in\mathbb{R}$ it holds that $|f(t,x,v)-f(t,x,w)| \leq L|v-w|$, the hypothesis that for all $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that $\mathbb{E}[|g(x+W_{T-t})| + \int_t^T|f(s,x+W_{s-t},u(s,x+W_{s-t}))|\,ds] < \infty$, and Lemma 2.3 establishes that for all $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$\big(\mathbb{E}\big[|u(t,x+W_t)|^q\big]\big)^{1/q} \tag{2.22}$$

$$
\leq \exp\big(L(T-t)\big)\left[\big(\mathbb{E}\big[|g(x+W_T)|^q\big]\big)^{1/q} + (T-t)^{(q-1)/q}\bigg(\int_t^T \mathbb{E}\big[|f(s,x+W_s,0)|^q\big]\,ds\bigg)^{1/q}\right]
$$
$$
\leq \exp(LT)\left[\big(\mathbb{E}\big[|g(x+W_T)|^q\big]\big)^{1/q} + T^{(q-1)/q}\bigg(\int_0^T \mathbb{E}\big[|f(s,x+W_s,0)|^q\big]\,ds\bigg)^{1/q}\right].
$$

Next observe that the hypothesis that for all $x \in \mathbb{R}^d$ it holds that $|g(x)| \leq \mathfrak{L}(1+\|x\|^p)$ and (2.21) show that for all $x \in \mathbb{R}^d$ it holds that

$$
\mathbb{E}\big[|g(x+W_T)|^q\big] \leq \mathbb{E}\big[\mathfrak{L}^q(1+\|x+W_T\|^p)^q\big] \leq \sup_{s\in[0,T]} \mathbb{E}\big[\mathfrak{L}^q(1+\|x+W_s\|^p)^q\big] < \infty. \tag{2.23}
$$

In addition, note that the hypothesis that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $|f(t,x,0)| \leq \mathfrak{L}(1+\|x\|^p)$ and (2.21) assure that for all $x \in \mathbb{R}^d$ it holds that

$$
\begin{split}
\bigg(\int_0^T \mathbb{E}\big[|f(s,x+W_s,0)|^q\big]\,ds\bigg)^{1/q} &\leq \bigg(\int_0^T \mathbb{E}\big[\mathfrak{L}^q(1+\|x+W_s\|^p)^q\big]\,ds\bigg)^{1/q} \\
&\leq \mathfrak{L}T^{1/q}\left[\sup_{s\in[0,T]} \big(\mathbb{E}\big[(1+\|x+W_s\|^p)^q\big]\big)^{1/q}\right] < \infty.
\end{split} \tag{2.24}
$$

Combining this, (2.22), and (2.23) proves that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$
\big(\mathbb{E}\big[|u(t,x+W_t)|^q\big]\big)^{1/q} \leq \mathfrak{L}(T+1)\exp(LT)\left[\sup_{s\in[0,T]} \big(\mathbb{E}\big[(1+\|x+W_s\|^p)^q\big]\big)^{1/q}\right] < \infty. \tag{2.25}
$$

The proof of Corollary 2.5 is thus complete. □

3 Full-history recursive multilevel Picard (MLP) approximations

In this section we introduce and provide the L^p-error analysis for MLP approximations for solutions of stochastic fixed-point equations. More specifically, we prove Corollary 3.15 below, which is a non-recursive L^p-error bound for MLP approximations for solutions of stochastic fixed-point equations. Our proof of Corollary 3.15 uses Lemma 3.14, which provides a potentially sharper L^p-error bound for MLP approximations for solutions of stochastic fixed-point equations. Our proof of Lemma 3.14, in turn, employs Lemma 3.10 and the elementary auxiliary results in Lemma 3.11, Lemma 3.12, and Lemma 3.13. Our proof of the recursive error bound in Lemma 3.10 employs Lemma 3.9. Our proof of Lemma 3.9, in turn, is based on Lemma 3.5 and the elementary Monte Carlo approximation results in Lemma 3.6, Corollary 3.7, and Corollary 3.8. Our proof of Lemma 3.5 uses Lemma 3.3 and Lemma 3.4, which are elementary results regarding the measurability and integrability of MLP approximations for solution of stochastic fixed-point equations, respectively.

Lemma 3.3 is, e.g., proved as Hutzenthaler et al. [30, Lemma 3.2]. Lemma 3.4 is, e.g., proved as Hutzenthaler et al. [30, Lemma 3.3]. Only for completeness we include in this section the detailed proofs of Lemma 3.3 and Lemma 3.4, respectively. Lemma 3.11 and Lemma 3.12 are well-known elementary results and we include their proofs for completeness, as well. The elementary result Lemma 3.13 is a slight generalization of the result in Hutzenthaler et al. [28, Lemma 3.11].

3.1 MLP approximations

Definition 3.1. *Let $p \in [2,\infty)$. Then we denote by $\mathfrak{K}_p \in \mathbb{R}$ the real number given by*

$$
\mathfrak{K}_p = \inf\left\{ c \in \mathbb{R} \colon \begin{bmatrix} \text{It holds for every probability space } (\Omega,\mathcal{F},\mathbb{P}) \text{ and every} \\ \text{random variable } X\colon \Omega \to \mathbb{R} \text{ with } \mathbb{E}[|X|] < \infty \text{ that} \\ \big(\mathbb{E}\big[|X-\mathbb{E}[X]|^p\big]\big)^{1/p} \leq c\big(\mathbb{E}\big[|X|^p\big]\big)^{1/p} \end{bmatrix} \right\}. \tag{3.1}
$$

Setting 3.2. *Let $d, m \in \mathbb{N}$, $T, L, \mathfrak{L}, p \in [0,\infty)$, $\mathfrak{p} \in [2,\infty)$, $\mathfrak{m} = \mathfrak{K}_{\mathfrak{p}}\sqrt{\mathfrak{p}-1}$, $\Theta = \bigcup_{n\in\mathbb{N}}\mathbb{Z}^n$, $f \in C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R}^d,\mathbb{R})$, let $F\colon C([0,T]\times\mathbb{R}^d,\mathbb{R}) \to C([0,T]\times\mathbb{R}^d,\mathbb{R})$, assume for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $w, \mathfrak{w} \in \mathbb{R}$, $v \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ that*

$$|f(t,x,w) - f(t,x,\mathfrak{w})| \le L|w-\mathfrak{w}|, \qquad \max\{|f(t,x,0)|, |g(x)|\} \le \mathfrak{L}(1+\|x\|^p), \tag{3.2}$$

and

$$(F(v))(t,x) = f(t,x,v(t,x)), \tag{3.3}$$

let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\mathfrak{u}^\theta\colon \Omega\to[0,1]$, $\theta\in\Theta$, be i.i.d. random variables, assume for all $\theta\in\Theta$, $r\in(0,1)$ that $\mathbb{P}(\mathfrak{u}^\theta \le r) = r$, let $\mathcal{U}^\theta\colon [0,T]\times\Omega\to[0,T]$, $\theta\in\Theta$, satisfy for all $t\in[0,T]$, $\theta\in\Theta$ that $\mathcal{U}^\theta_t = t + (T-t)\mathfrak{u}^\theta$, let $W^\theta\colon [0,T]\times\Omega\to\mathbb{R}^d$, $\theta\in\Theta$, be independent standard Brownian motions, assume that $(\mathcal{U}^\theta)_{\theta\in\Theta}$ and $(W^\theta)_{\theta\in\Theta}$ are independent, let $u\in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ satisfy for all $t\in[0,T]$, $x\in\mathbb{R}^d$ that $\mathbb{E}[|g(x+W^0_{T-t})| + \int_t^T |(F(u))(s,x+W^0_{s-t})|\,ds] < \infty$ and

$$u(t,x) = \mathbb{E}\big[g(x+W^0_{T-t})\big] + \int_t^T \mathbb{E}\big[(F(u))(s,x+W^0_{s-t})\big]\,ds, \tag{3.4}$$

and let $U^\theta_n\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $n\in\mathbb{Z}$, $\theta\in\Theta$, satisfy for all $n\in\mathbb{N}_0$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that

$$\begin{aligned} U^\theta_n(t,x) = {}& \frac{\mathbb{1}_{\mathbb{N}}(n)}{m^n}\left[\sum_{k=1}^{m^n} g(x + W^{(\theta,0,-k)}_{T-t})\right] \\ &+ \sum_{i=0}^{n-1}\frac{(T-t)}{m^{n-i}}\left[\sum_{k=1}^{m^{n-i}} \big(F(U^{(\theta,i,k)}_i) - \mathbb{1}_{\mathbb{N}}(i)F(U^{(\theta,-i,k)}_{i-1})\big)\big(\mathcal{U}^{(\theta,i,k)}_t, x + W^{(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}\big)\right] \end{aligned} \tag{3.5}$$

(cf. Definitions 2.4 and 3.1).

3.2 Measurability properties of MLP approximations

Lemma 3.3. *Assume Setting 3.2. Then*

(i) it holds for all $n\in\mathbb{N}_0$, $\theta\in\Theta$ that $U^\theta_n\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is a continuous random field,

(ii) it holds[1] for all $n\in\mathbb{N}_0$, $\theta\in\Theta$ that $\sigma(U^\theta_n) \subseteq \sigma((\mathcal{U}^{(\theta,\vartheta)})_{\vartheta\in\Theta}, (W^{(\theta,\vartheta)})_{\vartheta\in\Theta})$,

(iii) it holds for all $n\in\mathbb{N}_0$ that $(U^\theta_n)_{\theta\in\Theta}$, $(W^\theta)_{\theta\in\Theta}$, and $(\mathcal{U}^\theta)_{\theta\in\Theta}$ are independent,

(iv) it holds for all $n, m\in\mathbb{N}_0$, $i, k, \mathfrak{i}, \mathfrak{k}\in\mathbb{Z}$ with $(i,k)\neq(\mathfrak{i},\mathfrak{k})$ that $(U^{(\theta,i,k)}_n)_{\theta\in\Theta}$ and $(U^{(\theta,\mathfrak{i},\mathfrak{k})}_m)_{\theta\in\Theta}$ are independent, and

(v) it holds for all $n\in\mathbb{N}_0$ that $(U^\theta_n)_{\theta\in\Theta}$ are identically distributed random variables.

Proof of Lemma 3.3. We first prove item (i) by induction. For the base case $n=0$ note that (3.5) ensures that for all $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that $U^\theta_0(t,x) = 0$. This implies that for all $\theta\in\Theta$ it holds that $U^\theta_0\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is a continuous random field. This establishes item (i) in the base case $n=0$. For the induction step $\mathbb{N}_0 \ni (n-1) \dashrightarrow n \in\mathbb{N}$ let $n\in\mathbb{N}$ and assume that for every $k\in\{0,1,\ldots,n-1\}$, $\theta\in\Theta$ it holds that $U^\theta_k\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}^d$ is a continuous random field. This, the hypothesis that $f\in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, (3.3), and, e.g., Hutzenthaler et al. [30, Item (i) in Lemma 2.9] (applied for every $n\in\mathbb{N}_0$, $\theta\in\Theta$ with $d \curvearrowleft d$,

[1] Note that for every $\mathcal{A}\subseteq 2^\Omega$ it holds that $\sigma(\mathcal{A})$ is a sigma-algebra on Ω and note that for every $\mathcal{A}\subseteq 2^\Omega$ and every sigma-algebra $\mathcal{B}$ on Ω with $\mathcal{A}\subseteq\mathcal{B}$ it holds that $\sigma(\mathcal{A})\subseteq\mathcal{B}$.

$T \curvearrowleft T$, $(\Omega, \mathcal{F}, \mathbb{P}) \curvearrowleft (\Omega, \mathcal{F}, \mathbb{P})$, $F \curvearrowleft F$, $U \curvearrowleft U^\theta_k$ in the notation of [30, Item (i) of Lemma 2.9]) imply that for all $k \in \{0, 1, \ldots, n-1\}$, $\theta \in \Theta$ it holds that

$$[0,T] \times \mathbb{R}^d \times \Omega \ni (t,x,\omega) \mapsto \big[F\big([0,T] \times \mathbb{R}^d \ni (s,y) \mapsto U^\theta_k(s,y,\omega) \in \mathbb{R}\big)\big](t,x) \in \mathbb{R} \tag{3.6}$$

is a continuous random field. Combining this, the hypothesis that $g \in C(\mathbb{R}^d, \mathbb{R})$, the fact that for all $\theta \in \Theta$ it holds that $W^\theta \colon [0,T] \times \Omega \to \mathbb{R}^d$ and $\mathcal{U}^\theta \colon [0,T] \times \Omega \to [0,T]$ are continuous random fields, (3.5), Hutzenthaler et al. [31, Lemma 2.14], and Beck et al. [1, Lemma 2.4] proves that for all $\theta \in \Theta$ it holds that $U^\theta_n \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a continuous random field. Induction thus establishes item (i). Next note that (3.6), Beck et al. [1, Lemma 2.4], and item (i) assure that for all $n \in \mathbb{N}_0$, $\theta \in \Theta$ it holds that $F(U^\theta_n)$ is $(\mathcal{B}([0,T] \times \mathbb{R}^d) \otimes \sigma(U^\theta_n))/\mathcal{B}(\mathbb{R})$-measurable. This, (3.5), the fact that for all $\theta \in \Theta$ it holds that W^θ is $(\mathcal{B}([0,T]) \otimes \sigma(W^\theta))/\mathcal{B}(\mathbb{R}^d)$-measurable, the fact that for all $\theta \in \Theta$ it holds that $\mathcal{U}^\theta$ is $(\mathcal{B}([0,T]) \otimes \sigma(\mathfrak{u}^\theta))/\mathcal{B}([0,T])$-measurable, and induction on $\mathbb{N}_0$ prove item (ii). Moreover, observe that item (ii) and the fact that for all $\theta \in \Theta$ it holds that $(\mathcal{U}^{(\theta,\vartheta)})_{\vartheta \in \Theta}, (W^{(\theta,\vartheta)})_{\vartheta \in \Theta}$, W^θ, and $\mathfrak{u}^\theta$ are independent establish item (iii). Furthermore, note that item (ii) and the fact that for all $i, k, \mathfrak{i}, \mathfrak{k}, \in \mathbb{Z}$, $\theta \in \Theta$ with $(i,k) \neq (\mathfrak{i}, \mathfrak{k})$ it holds that $((\mathcal{U}^{(\theta,i,k,\vartheta)})_{\vartheta \in \Theta}, (W^{(\theta,i,k,\vartheta)})_{\vartheta \in \Theta})$ and $((\mathcal{U}^{(\theta,\mathfrak{i},\mathfrak{k},\vartheta)})_{\vartheta \in \Theta}, (W^{(\theta,\mathfrak{i},\mathfrak{k},\vartheta)})_{\vartheta \in \Theta})$ are independent establish item (iv). In addition, observe that the fact that (3.5) implies that for all $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $U^\theta_0(t,x) = 0$, the hypothesis that $(W^\theta)_{\theta \in \Theta}$ are independent standard Brownian motions, the hypothesis that $(\mathfrak{u}^\theta)_{\theta \in \Theta}$ are i.i.d. random variables, items (i), (ii), (iii), and (iv), Hutzenthaler et al. [30, Corollary 2.5], and induction on $\mathbb{N}_0$ establish item (v). The proof of Lemma 3.3 is thus complete. □

3.3 Integrability properties of MLP approximations

Lemma 3.4. *Assume Setting 3.2. Then it holds for all $n \in \mathbb{N}_0$, $\theta \in \Theta$, $s \in [0,T]$, $t \in [s,T]$, $x \in \mathbb{R}^d$ that*

$$\begin{aligned} &\mathbb{E}\big[|U^\theta_n(t, x + W^\theta_{t-s})|\big] + \mathbb{E}\big[|g(x + W^\theta_{t-s})|\big] + \mathbb{E}\big[|(F(U^\theta_n))(\mathcal{U}^\theta_t, x + W^\theta_{\mathcal{U}^\theta_t - t})|\big] \\ &+ \int_s^T \mathbb{E}\big[|U^\theta_n(r, x + W^\theta_{r-s})|\big]\,dr + \int_s^T \mathbb{E}\big[|(F(U^\theta_n))(r, x + W^\theta_{r-s})|\big]\,dr < \infty. \end{aligned} \tag{3.7}$$

Proof of Lemma 3.4. Throughout this proof let $x \in \mathbb{R}^d$ and assume without loss of generality that $T \in (0,\infty)$. Note that (3.2), the fact that for all $r, a, b \in [0,\infty)$ it holds that $(a+b)^r \leq 2^{\max\{r-1,0\}}(a^r + b^r)$, and the fact that for all $\theta \in \Theta$ it holds that $\mathbb{E}[\|W^\theta_T\|^p] < \infty$ assure that for all $s \in [0,T]$, $t \in [s,T]$, $\theta \in \Theta$ it holds that

$$\mathbb{E}\big[|g(x + W^\theta_{t-s})|\big] \leq \mathbb{E}\big[\mathfrak{L}(1 + \|x + W^\theta_{t-s}\|^p)\big] \leq \mathfrak{L}\big[1 + 2^{\max\{p-1,0\}}\big(\|x\|^p + \mathbb{E}\big[\|W^\theta_T\|^p\big]\big)\big] < \infty. \tag{3.8}$$

Next we claim that for all $n \in \mathbb{N}_0$, $s \in [0,T]$, $t \in [s,T]$, $\theta \in \Theta$ it holds that

$$\begin{aligned} &\mathbb{E}\big[|U^\theta_n(t, x + W^\theta_{t-s})|\big] + \mathbb{E}\big[|(F(U^\theta_n))(\mathcal{U}^\theta_t, x + W^\theta_{\mathcal{U}^\theta_t - t})|\big] \\ &+ \int_s^T \mathbb{E}\big[|U^\theta_n(r, x + W^\theta_{r-s})|\big]\,dr + \int_s^T \mathbb{E}\big[|(F(U^\theta_n))(r, x + W^\theta_{r-s})|\big]\,dr < \infty. \end{aligned} \tag{3.9}$$

We now prove (3.9) by induction on $n \in \mathbb{N}_0$. For the base case $n = 0$ note that the fact that (3.5) implies that for all $t \in [0,T]$, $\theta \in \Theta$ it holds that $U^\theta_0(t,x) = 0$ ensures that for all $s \in [0,T]$, $t \in [s,T]$ it holds that

$$\begin{aligned} &\mathbb{E}\big[|U^\theta_0(t, x + W^\theta_{t-s})|\big] + \mathbb{E}\big[|(F(U^\theta_0))(\mathcal{U}^\theta_t, x + W^\theta_{\mathcal{U}^\theta_t - t})|\big] \\ &+ \int_s^T \mathbb{E}\big[|U^\theta_0(r, x + W^\theta_{r-s})|\big]\,dr + \int_s^T \mathbb{E}\big[|(F(U^\theta_0))(r, x + W^\theta_{r-s})|\big]\,dr \\ &= \mathbb{E}\big[|(F(0))(\mathcal{U}^\theta_t, x + W^\theta_{\mathcal{U}^\theta_t - t})|\big] + \int_s^T \mathbb{E}\big[|(F(0))(r, x + W^\theta_{r-s})|\big]\,dr. \end{aligned} \tag{3.10}$$

In addition, observe that (3.2), (3.3), and (3.8) guarantee that for all $s \in [0,T]$, $t \in [s,T]$, $\theta \in \Theta$ it holds that

$$\begin{aligned} &\mathbb{E}\big[|(F(0))(\mathcal{U}_t^\theta, x + W^\theta_{\mathcal{U}_t^\theta - t})|\big] + \int_s^T \mathbb{E}\big[|(F(0))(r, x + W^\theta_{r-s})|\big]\, dr \\ &\leq \mathbb{E}\big[\mathfrak{L}\big(1 + \|x + W^\theta_{\mathcal{U}_t^\theta - t}\|^p\big)\big] + \int_s^T \mathbb{E}\big[\mathfrak{L}\big(1 + \|x + W^\theta_{r-s}\|^p\big)\big]\, dr \\ &\leq (T+1) \sup_{r\in[0,T]} \big(\mathbb{E}\big[\mathfrak{L}\big(1 + \|x + W^\theta_r\|^p\big)\big]\big) < \infty. \end{aligned} \tag{3.11}$$

Combining this and (3.10) establishes (3.9) in the base case $n = 0$. For the induction step $\mathbb{N}_0 \ni (n-1) \dashrightarrow n \in \mathbb{N}$ let $n \in \mathbb{N}$ and assume that for all $k \in \{0, 1, \ldots, n-1\}$, $s \in [0,T]$, $t \in [s,T]$, $\theta \in \Theta$ it holds that

$$\begin{aligned} &\mathbb{E}\big[|U^\theta_k(t, x + W^\theta_{t-s})|\big] + \mathbb{E}\big[|(F(U^\theta_k))(\mathcal{U}^\theta_t, x + W^\theta_{\mathcal{U}^\theta_t - t})|\big] \\ &+ \int_s^T \mathbb{E}\big[|U^\theta_k(r, x + W^\theta_{r-s})|\big]\, dr + \int_s^T \mathbb{E}\big[|(F(U^\theta_k))(r, x + W^\theta_{r-s})|\big]\, dr < \infty. \end{aligned} \tag{3.12}$$

Observe that (3.5) and the triangle inequality demonstrate that for all $s \in [0,T]$, $t \in [s,T]$, $\theta \in \Theta$ it holds that

$$\begin{aligned} \mathbb{E}\big[|U^\theta_n(t, x + W^\theta_{t-s})|\big] \leq \tfrac{\mathbb{1}_{\mathbb{N}}(n)}{m^n} &\Bigg[\sum_{i=1}^{m^n} \mathbb{E}\big[|g(x + W^\theta_{t-s} + W^{(\theta,0,-i)}_{T-t})|\big]\Bigg] \\ &+ \sum_{i=0}^{n-1} \tfrac{(T-t)}{m^{n-i}} \Bigg[\sum_{k=1}^{m^{n-i}} \mathbb{E}\big[|(F(U^{(\theta,i,k)}_i))(\mathcal{U}^{(\theta,i,k)}_t, x + W^\theta_{t-s} + W^{(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t})|\big] \\ &\quad + \mathbb{1}_{\mathbb{N}}(i)\, \mathbb{E}\big[|(F(U^{(\theta,-i,k)}_{i-1}))(\mathcal{U}^{(\theta,i,k)}_t, x + W^\theta_{t-s} + W^{(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t})|\big]\Bigg]. \end{aligned} \tag{3.13}$$

Next note that (3.8) and the fact that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions imply that for all $s \in [0,T]$, $t \in [s,T]$, $\theta \in \Theta$, $i \in \mathbb{Z}$ it holds that

$$\mathbb{E}\big[|g(x + W^\theta_{t-s} + W^{(\theta,0,i)}_{T-t})|\big] = \mathbb{E}\big[|g(x + W^\theta_{(t-s)+(T-t)})|\big] = \mathbb{E}\big[|g(x + W^\theta_{T-s})|\big] < \infty. \tag{3.14}$$

Furthermore, observe that (3.12), Lemma 3.3, the fact that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions, the fact that $(\mathcal{U}^\theta)_{\theta\in\Theta}$ are i.i.d. random variables, the hypothesis that $(W^\theta)_{\theta\in\Theta}$ and $(\mathcal{U}^\theta)_{\theta\in\Theta}$ are independent, the hypothesis that for all $\theta \in \Theta$, $r \in (0,1)$ it holds that $\mathbb{P}(\mathfrak{u}^\theta \leq r) = r$, Hutzenthaler et al. [31, Lemma 2.15], and Hutzenthaler et al. [31, Lemma 3.7] guarantee that for all $i \in \{0, 1, \ldots, n-1\}$, $k \in \mathbb{Z}$, $s \in [0,T]$, $t \in [s,T]$, $\theta \in \Theta$ it holds that

$$\begin{aligned} &(T-t)\, \mathbb{E}\big[|(F(U^{(\theta,i,k)}_i))(\mathcal{U}^{(\theta,i,k)}_t, x + W^\theta_{t-s} + W^{(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t})|\big] \\ &= \int_t^T \mathbb{E}\big[|(F(U^{(\theta,i,k)}_i))(r, x + W^\theta_{t-s} + W^{(\theta,i,k)}_{r-t})|\big]\, dr \\ &= \int_t^T \mathbb{E}\big[|(F(U^\theta_i))(r, x + W^\theta_{t-s} + W^\theta_{r-t})|\big]\, dr = \int_t^T \mathbb{E}\big[|(F(U^\theta_i))(r, x + W^\theta_{r-s})|\big]\, dr < \infty. \end{aligned} \tag{3.15}$$

Combining this, (3.12), (3.13), and (3.14) establishes that for all $s \in [0,T]$, $t \in [s,T]$, $\theta \in \Theta$ it holds that

$$\begin{aligned} \mathbb{E}\big[|U^\theta_n(t, x + W^\theta_{t-s})|\big] &\leq \Bigg(\sum_{i=0}^{n-1} \tfrac{1}{m^{n-i}} \Bigg[\sum_{k=1}^{m^{n-i}} \int_t^T \mathbb{E}\big[|(F(U^\theta_i))(r, x + W^\theta_{r-s})|\big]\, dr \\ &+ \mathbb{1}_{\mathbb{N}}(i) \int_t^T \mathbb{E}\big[|(F(U^\theta_{i-1}))(r, x + W^\theta_{r-s})|\big]\, dr\Bigg]\Bigg) + \frac{\mathbb{1}_{\mathbb{N}}(n)}{m^n} \Bigg[\sum_{i=1}^{m^n} \mathbb{E}\big[|g(x + W^\theta_{T-s})|\big]\Bigg] \end{aligned}$$

$$= \left[\sum_{i=0}^{n-1}\int_t^T \mathbb{E}\big[|(F(U_i^\theta))(r,x+W_{r-s}^\theta)|\big]\,dr \right. \tag{3.16}$$
$$\left. + \mathbb{1}_{\mathbb{N}}(i)\int_t^T \mathbb{E}\big[|(F(U_{i-1}^\theta))(r,x+W_{r-s}^\theta)|\big]\,dr\right] + \mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[|g(x+W_{T-s}^\theta)|\big]$$
$$\leq \mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[|g(x+W_{T-s}^\theta)|\big] + 2\left[\sum_{i=0}^{n-1}\int_t^T \mathbb{E}\big[|(F(U_i^\theta))(r,x+W_{r-s}^\theta)|\big]\,dr\right] < \infty.$$

This implies that for all $s\in[0,T]$, $\theta\in\Theta$ it holds that

$$\int_s^T \mathbb{E}\big[|U_n^\theta(r,x+W_{r-s}^\theta)|\big]\,dr \leq (T-s)\sup_{r\in[s,T]}\mathbb{E}\big[|U_n^\theta(r,x+W_{r-s}^\theta)|\big] \tag{3.17}$$
$$\leq (T-s)\left(\mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[|g(x+W_{T-s}^\theta)|\big] + 2\left[\sum_{i=0}^{n-1}\int_s^T \mathbb{E}\big[|(F(U_i^\theta))(r,x+W_{r-s}^\theta)|\big]\,dr\right]\right) < \infty.$$

Combining this, the triangle inequality, (3.2), (3.3), and (3.11) proves that for all $s\in[0,T]$, $\theta\in\Theta$ it holds that

$$\int_s^T \mathbb{E}\big[|(F(U_n^\theta))(r,x+W_{r-s}^\theta)|\big]\,dr$$
$$\leq \int_s^T \mathbb{E}\big[|(F(U_n^\theta)-F(0))(r,x+W_{r-s}^\theta)|\big]\,dr + \int_s^T \mathbb{E}\big[|(F(0))(r,x+W_{r-s}^\theta)|\big]\,dr \tag{3.18}$$
$$\leq L\int_s^T \mathbb{E}\big[|U_n^\theta(r,x+W_{r-s}^\theta)|\big]\,dr + \int_s^T \mathbb{E}\big[|(F(0))(r,x+W_{r-s}^\theta)|\big]\,dr < \infty.$$

This, (3.16), (3.17), and induction prove (3.9). Combining (3.9) with (3.8) therefore establishes (3.7). The proof of Lemma 3.4 is thus complete. □

3.4 Expectations of MLP approximations

Lemma 3.5. *Assume Setting 3.2. Then*

(i) it holds for all $n\in\mathbb{N}_0$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $(F(U_n^{(\theta,n,k)}))(\mathcal{U}_t^{(\theta,n,k)},x+W^{(\theta,n,k)}_{\mathcal{U}_t^{(\theta,n,k)}-t}) - \mathbb{1}_{\mathbb{N}}(n)(F(U_{n-1}^{(\theta,-n,k)}))(\mathcal{U}_t^{(\theta,n,k)},x+W^{(\theta,n,k)}_{\mathcal{U}_t^{(\theta,n,k)}-t})$, $k\in\mathbb{Z}$, are i.i.d. random variables,

(ii) it holds for all $n\in\mathbb{N}_0$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that

$$\mathbb{E}\big[|U_n^0(t,x)|\big] + \mathbb{E}\big[|g(x+W_{T-t}^{(0,0,-1)})|\big] \tag{3.19}$$
$$+\sum_{i=0}^{n-1}\mathbb{E}\big[|(F(U_i^{(0,i,1)}) - \mathbb{1}_{\mathbb{N}}(i)F(U_{i-1}^{(0,-i,1)}))(\mathcal{U}_t^{(0,i,1)},x+W^{(0,i,1)}_{\mathcal{U}_t^{(0,i,1)}-t})|\big] < \infty,$$

(iii) it holds for all $n\in\mathbb{N}_0$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that

$$\mathbb{E}\big[U_n^0(t,x)\big] = \mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[g\big(x+W_{T-t}^{(0,0,-1)}\big)\big] \tag{3.20}$$
$$+(T-t)\left[\sum_{i=0}^{n-1}\mathbb{E}\big[(F(U_i^{(0,i,1)}) - \mathbb{1}_{\mathbb{N}}(i)F(U_{i-1}^{(0,-i,1)}))(\mathcal{U}_t^{(0,i,1)},x+W^{(0,i,1)}_{\mathcal{U}_t^{(0,i,1)}-t})\big]\right],$$

and

(iv) it holds for all $n\in\mathbb{N}_0$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that

$$(T-t)\,\mathbb{E}\big[|(F(U_n^\theta) - \mathbb{1}_{\mathbb{N}}(n)F(U_{n-1}^\theta))(\mathcal{U}_t^\theta,x+W^\theta_{\mathcal{U}_t^\theta-t})|\big] \tag{3.21}$$
$$= \int_t^T \mathbb{E}\big[|(F(U_n^\theta) - \mathbb{1}_{\mathbb{N}}(n)F(U_{n-1}^\theta))(r,x+W_{r-t}^\theta)|\big]\,dr < \infty.$$

Proof of Lemma 3.5. Throughout this proof let $x \in \mathbb{R}^d$. Observe that Lemma 3.3, the hypothesis that $(\mathcal{U}^\theta)_{\theta\in\Theta}$ are i.i.d. random variables, the hypothesis that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions, Hutzenthaler et al. [30, Corollary 2.5], and Hutzenthaler et al. [30, Item (i) of Lemma 2.6] (applied for every $n \in \mathbb{N}_0$, $\theta \in \Theta$ with $d \curvearrowleft d$, $T \curvearrowleft T$, $(\Omega, \mathcal{F}, \mathbb{P}) \curvearrowleft (\Omega, \mathcal{F}, \mathbb{P})$, $F \curvearrowleft F$, $U \curvearrowleft U_n^\theta$ in the notation of [30, Item (i) of Lemma 2.6]) imply that for all $n \in \mathbb{N}_0$, $\theta \in \Theta$, $t \in [0, T]$ it holds that

$$\big(F(U_n^{(\theta,n,k)}) - \mathbb{1}_{\mathbb{N}}(n) F(U_{n-1}^{(\theta,-n,k)})\big)\big(\mathcal{U}_t^{(\theta,n,k)}, x + W^{(\theta,n,k)}_{\mathcal{U}_t^{(\theta,n,k)}-t}\big), \ k \in \mathbb{Z}, \tag{3.22}$$

are i.i.d. random variables. This establishes item (i). Next note that the triangle inequality, Lemma 3.3, Lemma 3.4, (3.3), (3.5), and the fact that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions guarantee that for all $n \in \mathbb{N}_0$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\mathbb{E}\big[|U_n^0(t,x)|\big] + \mathbb{E}\big[|g(x + W_{T-t}^{(0,0,-1)})|\big] \\
&+ \textstyle\sum_{i=0}^{n-1} \mathbb{E}\big[|(F(U_i^{(0,i,1)}) - \mathbb{1}_{\mathbb{N}}(i) F(U_{i-1}^{(0,-i,1)}))(\mathcal{U}_t^{(0,i,1)}, x + W^{(0,i,1)}_{\mathcal{U}_t^{(0,i,1)}-t})|\big] \\
&\le \mathbb{E}\big[|U_n^0(t,x)|\big] + \mathbb{E}\big[|g(x + W_{T-t}^{(0,0,-1)})|\big] + \textstyle\sum_{i=0}^{n-1} \mathbb{E}\big[|(F(U_i^{(0,i,1)}))(\mathcal{U}_t^{(0,i,1)}, x + W^{(0,i,1)}_{\mathcal{U}_t^{(0,i,1)}-t})|\big] \\
&\quad + \textstyle\sum_{i=0}^{n-1} \mathbb{1}_{\mathbb{N}}(i)\, \mathbb{E}\big[|(F(U_{i-1}^{(0,-i,1)}))(\mathcal{U}_t^{(0,i,1)}, x + W^{(0,i,1)}_{\mathcal{U}_t^{(0,i,1)}-t})|\big] \\
&\le \mathbb{E}\big[|U_n^0(t,x)|\big] + \mathbb{E}\big[|g(x + W_{T-t}^0)|\big] + 2\textstyle\sum_{i=0}^{n-1} \mathbb{E}\big[|(F(U_i^0))(\mathcal{U}_t^0, x + W^0_{\mathcal{U}_t^0-t})|\big] < \infty.
\end{aligned} \tag{3.23}$$

This establishes item (ii). Furthermore, observe that item (i), item (ii), Lemma 3.3, (3.3), (3.5), and the fact that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions ensure that for all $n \in \mathbb{N}_0$, $t \in [0, T]$ it holds that

$$\begin{aligned}
\mathbb{E}\big[U_n^0(t,x)\big] &= \frac{\mathbb{1}_{\mathbb{N}}(n)}{m^n}\left[\sum_{k=1}^{m^n} \mathbb{E}\big[g(x + W_{T-t}^{(0,0,-k)})\big]\right] \\
&+ \sum_{i=0}^{n-1} \tfrac{(T-t)}{m^{n-i}}\left[\sum_{k=1}^{m^{n-i}} \mathbb{E}\big[(F(U_i^{(0,i,k)}) - \mathbb{1}_{\mathbb{N}}(i) F(U_{i-1}^{(0,-i,k)}))(\mathcal{U}_t^{(0,i,k)}, x + W^{(0,i,k)}_{\mathcal{U}_t^{(0,i,k)}-t})\big]\right] \\
&= \mathbb{1}_{\mathbb{N}}(n)\, \mathbb{E}\big[g(x + W_{T-t}^{(0,0,-1)})\big] \\
&+ (T-t)\left[\sum_{i=0}^{n-1} \mathbb{E}\big[(F(U_i^{(0,i,1)}) - \mathbb{1}_{\mathbb{N}}(i) F(U_{i-1}^{(0,-i,1)}))(\mathcal{U}_t^{(0,i,1)}, x + W^{(0,i,1)}_{\mathcal{U}_t^{(0,i,1)}-t})\big]\right].
\end{aligned} \tag{3.24}$$

This establishes item (iii). In addition, observe that item (i), item (ii), Lemma 3.3, the fact that $(\mathcal{U}^\theta)_{\theta\in\Theta}$ are i.i.d. random variables, the hypothesis that for all $\theta \in \Theta$, $r \in (0,1)$ it holds that $\mathbb{P}(\mathfrak{u}^\theta \le r) = r$, and Hutzenthaler et al. [31, Lemma 3.7] demonstrate that for all $n \in \mathbb{N}_0$, $\theta \in \Theta$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&(T-t)\, \mathbb{E}\big[|(F(U_n^\theta) - \mathbb{1}_{\mathbb{N}}(n) F(U_{n-1}^\theta))(\mathcal{U}_t^\theta, x + W^\theta_{\mathcal{U}_t^\theta-t})|\big] \\
&= \int_t^T \mathbb{E}\big[|(F(U_n^\theta) - \mathbb{1}_{\mathbb{N}}(n) F(U_{n-1}^\theta))(r, x + W_{r-t}^\theta)|\big]\, dr < \infty.
\end{aligned} \tag{3.25}$$

This establishes item (iv). The proof of Lemma 3.5 is thus complete. □

3.5 Monte Carlo approximations

Lemma 3.6. *Let $p \in (2, \infty)$, $n \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_i \colon \Omega \to \mathbb{R}$, $i \in \{1, 2, \dots, n\}$, be i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Then it holds that*

$$\big(\mathbb{E}\big[|\mathbb{E}[X_1] - \tfrac{1}{n}(\textstyle\sum_{i=1}^n X_i)|^p\big]\big)^{1/p} \le \big[\tfrac{p-1}{n}\big]^{1/2}\big(\mathbb{E}\big[|X_1 - \mathbb{E}[X_1]|^p\big]\big)^{1/p}. \tag{3.26}$$

Proof of Lemma 3.6. First, observe that the hypothesis that for all $i \in \{1, 2, \dots, n\}$ it holds that $X_i\colon \Omega \to \mathbb{R}$ are i.i.d. random variables assures that

$$\mathbb{E}\big[|\mathbb{E}[X_1] - \tfrac{1}{n}(\textstyle\sum_{i=1}^n X_i)|^p\big] = \mathbb{E}\big[|\tfrac{1}{n}(\textstyle\sum_{i=1}^n(\mathbb{E}[X_1] - X_i))|^p\big] = n^{-p}\,\mathbb{E}\big[|\textstyle\sum_{i=1}^n(\mathbb{E}[X_i] - X_i)|^p\big]. \tag{3.27}$$

Combining this, the fact that for all $i \in \{1, 2, \dots, n\}$ it holds that $X_i\colon \Omega \to \mathbb{R}$ are i.i.d. random variables, and, e.g., [43, Theorem 2.1] (applied with $p \curvearrowleft p$, $(S_i)_{i\in\{0,1,\dots,n\}} \curvearrowleft (\sum_{k=1}^i(\mathbb{E}[X_k] - X_k))_{i\in\{0,1,\dots,n\}}$, $(X_i)_{i\in\{1,2,\dots,n\}} \curvearrowleft (\mathbb{E}[X_i] - X_i)_{i\in\{1,2,\dots,n\}}$ in the notation of [43, Theorem 2.1]) ensures that

$$\begin{aligned}
&\big(\mathbb{E}\big[|\mathbb{E}[X_1] - \tfrac{1}{n}(\textstyle\sum_{i=1}^n X_i)|^p\big]\big)^{2/p} = \tfrac{1}{n^2}\big(\mathbb{E}\big[|\textstyle\sum_{i=1}^n(\mathbb{E}[X_i] - X_i)|^p\big]\big)^{2/p} \\
&\le \tfrac{(p-1)}{n^2}\Big[\textstyle\sum_{i=1}^n\big(\mathbb{E}\big[|\mathbb{E}[X_i] - X_i|^p\big]\big)^{2/p}\Big] = \tfrac{(p-1)}{n^2}\Big[n\big(\mathbb{E}\big[|\mathbb{E}[X_1] - X_1|^p\big]\big)^{2/p}\Big] \\
&= \tfrac{(p-1)}{n}\big(\mathbb{E}\big[|\mathbb{E}[X_1] - X_1|^p\big]\big)^{2/p}.
\end{aligned} \tag{3.28}$$

The proof of Lemma 3.6 is thus complete. □

Corollary 3.7. *Let $p \in [2, \infty)$, $n \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_i\colon \Omega \to \mathbb{R}$, $i \in \{1, 2, \dots, n\}$, be i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Then it holds that*

$$\big(\mathbb{E}\big[|\mathbb{E}[X_1] - \tfrac{1}{n}(\textstyle\sum_{i=1}^n X_i)|^p\big]\big)^{1/p} \le \big[\tfrac{p-1}{n}\big]^{1/2}\big(\mathbb{E}\big[|X_1 - \mathbb{E}[X_1]|^p\big]\big)^{1/p}. \tag{3.29}$$

Proof of Corollary 3.7. Observe that, e.g., Grohs et al. [21, Lemma 2.3] and Lemma 3.6 establish (3.29). The proof of Corollary 3.7 is thus complete. □

Corollary 3.8. *Let $p \in [2, \infty)$, $n \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_i\colon \Omega \to \mathbb{R}$, $i \in \{1, 2, \dots, n\}$, be i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Then*

$$\big(\mathbb{E}\big[|\mathbb{E}[X_1] - \tfrac{1}{n}(\textstyle\sum_{i=1}^n X_i)|^p\big]\big)^{1/p} \le \tfrac{\mathfrak{K}_p\sqrt{p-1}}{n^{1/2}}\big(\mathbb{E}\big[|X_1|^p\big]\big)^{1/p} \tag{3.30}$$

(cf. Definition 3.1).

Proof of Corollary 3.8. Note that Definition 3.1 and Corollary 3.7 demonstrate that (3.30) holds. The proof of Corollary 3.8 is thus complete. □

3.6 Recursive error bounds for MLP approximations

Lemma 3.9. *Assume Setting 3.2. Then it holds for all $n \in \mathbb{N}_0$, $t \in [0, T]$, $x \in \mathbb{R}^d$ that*

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\big|U_n^0(t, x + W_t^0) - \mathbb{E}\big[U_n^0(t, x + W_t^0)\big]\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \\
&\le \frac{\mathbb{1}_{\mathbb{N}}(n)\mathfrak{m}}{m^{n/2}}\left[\Big(\mathbb{E}\Big[\big|g(x + W_T^0)\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} + (T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\Big(\int_t^T \mathbb{E}\Big[\big|f(s, x + W_s^0, 0)\big|^{\mathfrak{p}}\Big]\,ds\Big)^{1/\mathfrak{p}}\right] \\
&+ \sum_{i=0}^{n-1}\frac{L(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\left[\big(\mathbb{1}_{(0,n)}(i) + \mathbb{1}_{[0,n-1)}(i)\,m^{1/2}\big)\Big(\int_t^T \mathbb{E}\Big[\big|(U_i^0 - u)(s, x + W_s^0)\big|^{\mathfrak{p}}\Big]\,ds\Big)^{1/\mathfrak{p}}\right].
\end{aligned} \tag{3.31}$$

Proof of Lemma 3.9. Throughout this proof let $\mathbf{G}_k\colon [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $k \in \mathbb{Z}$, satisfy for all $k \in \mathbb{Z}$, $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$\mathbf{G}_k(t, x) = g(x + W_{T-t}^{(0,0,-k)}) \tag{3.32}$$

and let $\mathbf{F}_{n,i}^{j,k}\colon [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $n, i, j, k \in \mathbb{Z}$, satisfy for all $n \in \mathbb{N}$, $i \in \{0, 1, \dots, n-1\}$, $j, k \in \mathbb{Z}$, $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$\mathbf{F}_{n,i}^{j,k}(t, x) = (F(U_i^{(0,j,k)}) - \mathbb{1}_{\mathbb{N}}(i)F(U_{i-1}^{(0,-j,k)}))(\mathcal{U}_t^{(0,j,k)}, x + W_{\mathcal{U}_t^{(0,j,k)}-t}^{(0,j,k)}). \tag{3.33}$$

Observe that the hypothesis that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions and the hypothesis that $g \in C(\mathbb{R}^d, \mathbb{R})$ assure that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $(\mathbf{G}_k(t,x))_{k\in\mathbb{Z}}$ are i.i.d. random variables. This and Corollary 3.8 (applied for every $n \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ with $p \curvearrowleft \mathfrak{p}$, $n \curvearrowleft m^n$, $(X_k)_{k\in\{1,2,\dots,m^n\}} \curvearrowleft (\mathbf{G}_k(t,x))_{k\in\{1,2,\dots,m^n\}}$ in the notation of Corollary 3.8) ensure that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\Big(\mathbb{E}\Big[\big|\tfrac{1}{m^n}\big[\textstyle\sum_{k=1}^{m^n}\mathbf{G}_k(t,x)\big] - \mathbb{E}\big[\mathbf{G}_1(t,x)\big]\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \leq \tfrac{\mathfrak{m}}{m^{n/2}}\big(\mathbb{E}\big[|\mathbf{G}_1(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}. \tag{3.34}$$

Next note that item (i) of Lemma 3.5 and Corollary 3.8 (applied for every $n \in \mathbb{N}$, $i \in \{0,1,\dots,n-1\}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ with $p \curvearrowleft \mathfrak{p}$, $n \curvearrowleft m^{n-i}$, $(X_k)_{k\in\{1,2,\dots,m^{n-i}\}} \curvearrowleft (\mathbf{F}^{i,k}_{n,i}(t,x))_{k\in\{1,2,\dots,m^{n-i}\}}$ in the notation of Corollary 3.8) demonstrate that for all $n \in \mathbb{N}$, $i \in \{0,1,\dots,n-1\}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\Big(\mathbb{E}\Big[\big|\tfrac{1}{m^{n-i}}\big[\textstyle\sum_{k=1}^{m^{n-i}}\mathbf{F}^{i,k}_{n,i}(t,x)\big] - \mathbb{E}\big[\mathbf{F}^{i,1}_{n,i}(t,x)\big]\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \leq \tfrac{\mathfrak{m}}{m^{(n-i)/2}}\big(\mathbb{E}\big[|\mathbf{F}^{i,1}_{n,i}(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}. \tag{3.35}$$

Combining this, (3.5), (3.32), (3.33), (3.34), item (iii) of Lemma 3.5, and the triangle inequality implies that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\big|U^0_n(t,x) - \mathbb{E}\big[U^0_n(t,x)\big]\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \\
&= \Big(\mathbb{E}\Big[\big|\big(\tfrac{\mathbb{1}_{\mathbb{N}}(n)}{m^n}\big[\textstyle\sum_{k=1}^{m^n}\mathbf{G}_k(t,x)\big] + \sum_{i=0}^{n-1}\tfrac{(T-t)}{m^{n-i}}\big[\sum_{k=1}^{m^{n-i}}\mathbf{F}^{i,k}_{n,i}(t,x)\big]\big) \\
&\quad - \big(\mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[\mathbf{G}_1(t,x)\big] + \textstyle\sum_{i=0}^{n-1}(T-t)\,\mathbb{E}\big[\mathbf{F}^{i,1}_{n,i}(t,x)\big]\big)\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \\
&\leq \mathbb{1}_{\mathbb{N}}(n)\Big(\mathbb{E}\Big[\big|\tfrac{1}{m^n}\big[\textstyle\sum_{k=1}^{m^n}\mathbf{G}_k(t,x)\big] - \mathbb{E}\big[\mathbf{G}_1(t,x)\big]\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \\
&\quad + \textstyle\sum_{i=0}^{n-1}(T-t)\Big(\mathbb{E}\Big[\big|\tfrac{1}{m^{n-i}}\big[\sum_{k=1}^{m^{n-i}}\mathbf{F}^{i,k}_{n,i}(t,x)\big] - \mathbb{E}\big[\mathbf{F}^{i,1}_{n,i}(t,x)\big]\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \\
&\leq \tfrac{\mathbb{1}_{\mathbb{N}}(n)\mathfrak{m}}{m^{n/2}}\big(\mathbb{E}\big[|\mathbf{G}_1(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} + \sum_{i=0}^{n-1}\tfrac{(T-t)\mathfrak{m}}{m^{(n-i)/2}}\big(\mathbb{E}\big[|\mathbf{F}^{i,1}_{n,i}(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}.
\end{aligned} \tag{3.36}$$

Moreover, observe that (3.33) and items (i) and (iv) of Lemma 3.5 assure that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\sum_{i=0}^{n-1}\tfrac{(T-t)\mathfrak{m}}{m^{(n-i)/2}}\big(\mathbb{E}\big[|\mathbf{F}^{i,1}_{n,i}(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} = \sum_{i=0}^{n-1}\tfrac{(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\big((T-t)\,\mathbb{E}\big[|\mathbf{F}^{i,1}_{n,i}(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} \\
&= \sum_{i=0}^{n-1}\tfrac{(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\bigg(\int_t^T \mathbb{E}\Big[\big|\big(F(U^{(0,i,1)}_i) - \mathbb{1}_{\mathbb{N}}(i)F(U^{0,-i,1)}_{i-1})\big)\big(s, x + W^{(0,i,1)}_{s-t}\big)\big|^{\mathfrak{p}}\Big]\,ds\bigg)^{1/\mathfrak{p}}.
\end{aligned} \tag{3.37}$$

Furthermore, note that (3.3), (3.5), and the triangle inequality guarantee that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\sum_{i=0}^{n-1}\tfrac{(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\bigg(\int_t^T \mathbb{E}\Big[\big|\big(F(U^{(0,i,1)}_i) - \mathbb{1}_{\mathbb{N}}(i)F(U^{0,-i,1)}_{i-1})\big)\big(s, x + W^{(0,i,1)}_{s-t}\big)\big|^{\mathfrak{p}}\Big]\,ds\bigg)^{1/\mathfrak{p}} \\
&\leq \tfrac{\mathbb{1}_{\mathbb{N}}(n)(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{n/2}}\bigg(\int_t^T \mathbb{E}\Big[\big|f(s, x + W^{(0,0,1)}_{s-t}, 0)\big|^{\mathfrak{p}}\Big]\,ds\bigg)^{1/\mathfrak{p}} \\
&\quad + \sum_{i=1}^{n-1}\tfrac{(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\Bigg[\bigg(\int_t^T \mathbb{E}\Big[\big|\big(F(U^{(0,i,1)}_i) - F(u)\big)\big(s, x + W^{(0,i,1)}_{s-t}\big)\big|^{\mathfrak{p}}\Big]\,ds\bigg)^{1/\mathfrak{p}} \\
&\quad + \bigg(\int_t^T \mathbb{E}\Big[\big|\big(F(u) - F(U^{(0,-i,1)}_{i-1})\big)\big(s, x + W^{(0,i,1)}_{s-t}\big)\big|^{\mathfrak{p}}\Big]\,ds\bigg)^{1/\mathfrak{p}}\Bigg].
\end{aligned} \tag{3.38}$$

Combining this, Lemma 3.3, (3.2), (3.3), (3.32), (3.36), (3.37), and the fact that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions demonstrates that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\big|U_n^0(t,x) - \mathbb{E}\big[U_n^0(t,x)\big]\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \leq \tfrac{\mathbb{1}_{\mathbb{N}}(n)\mathfrak{m}}{m^{n/2}}\big(\mathbb{E}\big[|\mathbf{G}_1(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} + \sum_{i=0}^{n-1}\tfrac{(T-t)\mathfrak{m}}{m^{(n-i)/2}}\big(\mathbb{E}\big[|\mathbf{F}_{n,i}^{i,1}(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} \\
&\leq \tfrac{\mathbb{1}_{\mathbb{N}}(n)\mathfrak{m}}{m^{n/2}}\left[\Big(\mathbb{E}\Big[\big|g(x+W^1_{T-t})\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} + (T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\int_t^T \mathbb{E}\Big[\big|f(s,x+W^{(0,0,1)}_{s-t},0)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right] \\
&\quad+ \sum_{i=1}^{n-1}\tfrac{L(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\left[\left(\int_t^T \mathbb{E}\Big[\big|\big(U_i^{(0,i,1)}-u\big)\big(s,x+W^{(0,i,1)}_{s-t}\big)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right. \\
&\quad\left.+ \left(\int_t^T \mathbb{E}\Big[\big|\big(u-U_{i-1}^{(0,-i,1)}\big)\big(s,x+W^{(0,i,1)}_{s-t}\big)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right] \\
&= \tfrac{\mathbb{1}_{\mathbb{N}}(n)\mathfrak{m}}{m^{n/2}}\left[\Big(\mathbb{E}\Big[\big|g(x+W^0_{T-t})\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} + (T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\int_t^T \mathbb{E}\Big[\big|f(s,x+W^0_{s-t},0)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right] \\
&\quad+ \sum_{i=0}^{n-1}\tfrac{L(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\left[\big(\mathbb{1}_{(0,n)}(i) + \mathbb{1}_{[0,n-1)}(i)\,m^{1/2}\big)\left(\int_t^T \mathbb{E}\Big[\big|\big(U_i^0-u\big)\big(s,x+W^0_{s-t}\big)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right].
\end{aligned} \tag{3.39}$$

This and the fact that W^0 has independent increments ensure that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\big|U_n^0(t,x+W_t^0) - \mathbb{E}\big[U_n^0(t,x+W_t^0)\big]\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \\
&\leq \tfrac{\mathbb{1}_{\mathbb{N}}(n)\mathfrak{m}}{m^{n/2}}\left[\Big(\mathbb{E}\Big[\big|g(x+W^0_T)\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} + (T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\int_t^T \mathbb{E}\Big[\big|f(s,x+W^0_s,0)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right] \\
&\quad+ \sum_{i=0}^{n-1}\tfrac{L(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\left[\big(\mathbb{1}_{(0,n)}(i) + \mathbb{1}_{[0,n-1)}(i)\,m^{1/2}\big)\left(\int_t^T \mathbb{E}\Big[\big|\big(U_i^0-u\big)\big(s,x+W^0_{s-t}\big)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right].
\end{aligned} \tag{3.40}$$

The proof of Lemma 3.9 is thus complete. ☐

Lemma 3.10. *Assume Setting 3.2. Then it holds for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that*

$$\begin{aligned}
&\Big(\mathbb{E}\Big[\big|U_n^0(t,x+W_t^0) - u(t,x+W_t^0)\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \\
&\leq \frac{\mathfrak{m}\exp(L(T-t))}{m^{n/2}}\left[\Big(\mathbb{E}\Big[\big|g(x+W^0_T)\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} + (T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\int_t^T \mathbb{E}\Big[\big|f(s,x+W^0_s,0)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right] \\
&\quad+ \sum_{i=0}^{n-1}\frac{L(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\left[\big(\mathbb{1}_{(0,n)}(i) + m^{1/2}\big)\left(\int_t^T \mathbb{E}\Big[\big|(U_i^0-u)(s,x+W^0_s)\big|^{\mathfrak{p}}\Big]\,ds\right)^{1/\mathfrak{p}}\right].
\end{aligned} \tag{3.41}$$

Proof of Lemma 3.10. First, observe that Lemma 3.4, Corollary 2.5, and the triangle inequality ensure that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\big(\mathbb{E}\big[|U_n^0(t,x) - u(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} \\
&\leq \big(\mathbb{E}\big[|U_n^0(t,x) - \mathbb{E}[U_n^0(t,x)]|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} + \big(\mathbb{E}\big[|\mathbb{E}[U_n^0(t,x)] - u(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}.
\end{aligned} \tag{3.42}$$

Next note that items (ii), (iii), and (iv) of Lemma 3.5, the fact that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions, and (3.5) demonstrate that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\mathbb{E}\big[U_n^0(t,x)\big] = \mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[g(x+W^{(0,0,-1)}_{T-t})\big]$$

$$+ (T-t)\Bigg[\sum_{i=0}^{n-1}\mathbb{E}\big[(F(U_i^{(0,i,1)}) - \mathbb{1}_{\mathbb{N}}(i)F(U_{i-1}^{(0,-i,1)}))(\mathcal{U}_t^{(0,i,1)}, x + W^{(0,i,1)}_{\mathcal{U}_t^{(0,i,1)}-t})\big]\Bigg] \tag{3.43}$$
$$= \mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[g(x+W^0_{T-t})\big] + \Bigg[\sum_{i=0}^{n-1}\int_t^T \mathbb{E}\big[(F(U_i^{(0,i,1)}) - \mathbb{1}_{\mathbb{N}}(i)F(U_{i-1}^{(0,-i,1)}))(s, x+W^{(0,i,1)}_{s-t})\big]\,ds\Bigg].$$

In addition, observe that (3.3), the fact that $(W^\theta)_{\theta\in\Theta}$ are independent standard Brownian motions, the fact that $(\mathcal{U}^\theta)_{\theta\in\Theta}$ are i.i.d. random variables, items (iii) and (v) of Lemma 3.3, and [30, Lemma 2.2] prove that for all $i\in\mathbb{N}_0$, $t\in[0,T]$, $s\in[t,T]$, $x\in\mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathbb{E}\big[(F(U_i^{(0,i,1)}) - \mathbb{1}_{\mathbb{N}}(i)F(U_{i-1}^{(0,-i,1)}))(s, x+W^{(0,i,1)}_{s-t})\big]\\ &= \mathbb{E}\big[(F(U_i^{(0,i,1)}))(s,x+W^{(0,i,1)}_{s-t})\big] - \mathbb{1}_{\mathbb{N}}(i)\,\mathbb{E}\big[(F(U_{i-1}^{(0,-i,1)}))(s,x+W^{(0,i,1)}_{s-t})\big]\\ &= \mathbb{E}\big[(F(U_i^0))(s,x+W^0_{s-t})\big] - \mathbb{1}_{\mathbb{N}}(i)\,\mathbb{E}\big[(F(U_{i-1}^0))(s,x+W^0_{s-t})\big].\end{aligned} \tag{3.44}$$

Combining this, Lemma 3.4, and (3.43) yields that for all $n\in\mathbb{N}_0$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathbb{E}\big[U_n^0(t,x)\big] = \mathbb{1}_{\mathbb{N}}(n)\,\mathbb{E}\big[g(x+W^0_{T-t})\big]\\ &+ \Bigg[\sum_{i=0}^{n-1}\int_t^T\Big(\mathbb{E}\big[(F(U_i^0))(s,x+W^0_{s-t})\big] - \mathbb{1}_{\mathbb{N}}(i)\,\mathbb{E}\big[(F(U_{i-1}^0))(s,x+W^0_{s-t})\big]\Big)\,ds\Bigg]\\ &= \mathbb{1}_{\mathbb{N}}(n)\Bigg[\mathbb{E}\big[g(x+W^0_{T-t})\big] + \int_t^T \mathbb{E}\big[(F(U^0_{n-1}))(s,x+W^0_{s-t})\big]\,ds\Bigg].\end{aligned} \tag{3.45}$$

This and (3.4) show that for all $n\in\mathbb{N}_0$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$u(t,x) - \mathbb{E}\big[U_n^0(t,x)\big] = \begin{cases} u(t,x) & : n=0\\ \int_t^T \mathbb{E}\big[(F(u)-F(U^0_{n-1}))(s,x+W^0_{s-t})\big]\,ds & : n\in\mathbb{N}\end{cases}. \tag{3.46}$$

This, (3.2), (3.3), Corollary 2.5, the triangle inequality, Jensen's inequality, Fubini's theorem, and the fact that W^0 has independent increments assure that for all $n\in\mathbb{N}_0$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$\begin{aligned}&\big(\mathbb{E}\big[|\mathbb{E}[U_n^0(t,x+W_t^0)] - u(t,x+W_t^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}\\ &\le \mathbb{1}_{\{0\}}(n)\big(\mathbb{E}\big[|u(t,x+W_t^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} + \mathbb{1}_{\mathbb{N}}(n)\Bigg(\mathbb{E}\Bigg[\bigg|\int_t^T \mathbb{E}\big[(F(u)-F(U^0_{n-1}))(s,x+W_s^0)\big]\,ds\bigg|^{\mathfrak{p}}\Bigg]\Bigg)^{1/\mathfrak{p}}\\ &\le \mathbb{1}_{\{0\}}(n)\big(\mathbb{E}\big[|u(t,x+W_t^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} + \mathbb{1}_{\mathbb{N}}(n)\int_t^T\big(\mathbb{E}\big[|(F(u)-F(U^0_{n-1}))(s,x+W_s^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}\,ds\\ &\le \mathbb{1}_{\{0\}}(n)\big(\mathbb{E}\big[|u(t,x+W_t^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} + \mathbb{1}_{\mathbb{N}}(n)\,L\int_t^T\big(\mathbb{E}\big[|(u-U^0_{n-1})(s,x+W_s^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}\,ds.\end{aligned} \tag{3.47}$$

Next observe that Hölder's inequality ensures that for all $n\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$\begin{aligned}&\int_t^T\big(\mathbb{E}\big[|(u-U^0_{n-1})(s,x+W_s^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}\,ds = \Bigg(\bigg[\int_t^T\big(\mathbb{E}\big[|(u-U^0_{n-1})(s,x+W_s^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}\,ds\bigg]^{\mathfrak{p}}\Bigg)^{1/\mathfrak{p}}\\ &\le \bigg((T-t)^{\mathfrak{p}-1}\int_t^T \mathbb{E}\big[|(u-U^0_{n-1})(s,x+W_s^0)|^{\mathfrak{p}}\big]\,ds\bigg)^{1/\mathfrak{p}}\\ &= (T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\bigg(\int_t^T \mathbb{E}\big[|(u-U^0_{n-1})(s,x+W_s^0)|^{\mathfrak{p}}\big]\,ds\bigg)^{1/\mathfrak{p}}.\end{aligned} \tag{3.48}$$

Combining this, Lemma 2.3, and (3.47) demonstrates that for all $n\in\mathbb{N}_0$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that

$$\Big(\mathbb{E}\Big[\big|\mathbb{E}\big[U_n^0(t,x+W_t^0)\big] - u(t,x+W_t^0)\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}}$$

$$\leq \mathbb{1}_{\{0\}}(n)\exp\bigl(L(T-t)\bigr)\Bigl[\bigl(\mathbb{E}\bigl[|g(x+W_T)|^{\mathfrak{p}}\bigr]\bigr)^{1/\mathfrak{p}} + (T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\Bigl(\int_t^T \mathbb{E}\bigl[|f(s,x+W_s,0)|^{\mathfrak{p}}\bigr]\,ds\Bigr)^{1/\mathfrak{p}}\Bigr]$$
$$+ \mathbb{1}_{\mathbb{N}}(n)\,L(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\Bigl(\int_t^T \mathbb{E}\bigl[|(u-U^0_{n-1})(s,x+W^0_s)|^{\mathfrak{p}}\bigr]\,ds\Bigr)^{1/\mathfrak{p}}. \tag{3.49}$$

This, Lemma 3.9, (3.42), and the fact that $\mathfrak{m} \in [1,\infty)$ guarantee that for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\bigl(\mathbb{E}\bigl[|U^0_n(t,x+W^0_t) - u(t,x+W^0_t)|^{\mathfrak{p}}\bigr]\bigr)^{1/\mathfrak{p}} \\
&\leq \frac{\mathfrak{m}\exp(L(T-t))}{m^{n/2}}\Bigl[\bigl(\mathbb{E}\bigl[|g(x+W^0_T)|^{\mathfrak{p}}\bigr]\bigr)^{1/\mathfrak{p}} + (T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\Bigl(\int_t^T \mathbb{E}\bigl[|f(s,x+W^0_s,0)|^{\mathfrak{p}}\bigr]\,ds\Bigr)^{1/\mathfrak{p}}\Bigr] \\
&\quad+ \sum_{i=0}^{n-1}\frac{L(T-t)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\Bigl[\bigl(\mathbb{1}_{(0,n)}(i) + m^{1/2}\bigr)\Bigl(\int_t^T \mathbb{E}\bigl[|(U^0_{i-1}-u)(s,x+W^0_s)|^{\mathfrak{p}}\bigr]\,ds\Bigr)^{1/\mathfrak{p}}\Bigr].
\end{aligned} \tag{3.50}$$

The proof of Lemma 3.10 is thus complete. □

Lemma 3.11. *It holds for all $n \in \mathbb{N}$ that*

$$\Bigl[\frac{n}{3}\Bigr]^n \leq \Bigl[\frac{n}{e}\Bigr]^n < e\Bigl[\frac{n}{e}\Bigr]^n < \frac{e}{2^{1/2}}\Bigl[\frac{n}{e}\Bigr]^{n+1/2} \leq n! \leq e\Bigl[\frac{n}{e}\Bigr]^{n+1/2} < e\Bigl[\frac{n+1}{e}\Bigr]^{n+1} \tag{3.51}$$

and

$$n^n \leq 2^{-1/2}e^{1/2}n^{n+1/2} \leq (n!)e^n \leq e^{1/2}n^{n+1/2} \leq (n+1)^{(n+1)}. \tag{3.52}$$

Proof of Lemma 3.11. Throughout this proof let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy for all $x \in [2,\infty)$ that $f(x) = (x-\frac{1}{2})(\ln(x)-\ln(x-1))$. Observe that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned}
\ln(n!) &= \ln\bigl(n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1\bigr) = \sum_{k=1}^n \ln(k) \\
&= \sum_{k=2}^n \Bigl[\int_{k-1}^k \ln(x)\,dx + \Bigl(\ln(k) - \int_{k-1}^k \ln(x)\,dx\Bigr)\Bigr].
\end{aligned} \tag{3.53}$$

In addition, note that for all $k \in \mathbb{N}$ it holds that

$$\begin{aligned}
\ln(k) - \int_{k-1}^k \ln(x)\,dx &= \ln(k) - \bigl[\bigl(k\ln(k)-k\bigr) - \bigl((k-1)\ln(k-1)-(k-1)\bigr)\bigr] \\
&= 1 - (k-1)\bigl(\ln(k)-\ln(k-1)\bigr).
\end{aligned} \tag{3.54}$$

This and (3.53) yield that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned}
\ln(n!) &= \sum_{k=2}^n \int_{k-1}^k \ln(x)\,dx + \sum_{k=2}^n \bigl[1-(k-1)\bigl(\ln(k)-\ln(k-1)\bigr)\bigr] \\
&= \int_1^n \ln(x)\,dx + \sum_{k=2}^n \bigl[1-(k-1)\bigl(\ln(k)-\ln(k-1)\bigr)\bigr] \\
&= n\ln(n) - n + 1 + \tfrac{1}{2}\sum_{k=2}^n \bigl(\ln(k)-\ln(k-1)\bigr) + \textstyle\sum_{k=2}^n\bigl[1-(k-\frac{1}{2})\bigl(\ln(k)-\ln(k-1)\bigr)\bigr] \\
&= (n+\tfrac{1}{2})\ln(n) - n + 1 + \sum_{k=2}^n [1-f(k)].
\end{aligned} \tag{3.55}$$

Next observe that the fact that for all $x \in [2,\infty)$ it holds that $f(x) = (x-\frac{1}{2})(\ln(x)-\ln(x-1))$ implies that for all $x \in [2,\infty)$ it holds that

$$f'(x) = \bigl(\ln(x)-\ln(x-1)\bigr) + (x-\tfrac{1}{2})\bigl(\tfrac{1}{x} - \tfrac{1}{x-1}\bigr). \tag{3.56}$$

This ensures that for all $x \in [2,\infty)$ it holds that

$$\begin{aligned} f''(x) &= 2\big(\tfrac{1}{x} - \tfrac{1}{x-1}\big) - (x - \tfrac{1}{2})\big(\tfrac{1}{x^2} - \tfrac{1}{(x-1)^2}\big) \\ &= \tfrac{2}{x^2(x-1)^2}\Big[2\big(x(x-1)^2 - x^2(x-1)\big) - (x-\tfrac{1}{2})\big((x-1)^2 - x^2\big)\Big] \\ &= \tfrac{2}{x^2(x-1)^2}\big[-2x(x-1) + (x-\tfrac{1}{2})(2x-1)\big] = \tfrac{1}{x^2(x-1)^2} > 0. \end{aligned} \tag{3.57}$$

Combining this and (3.56) shows that for all $x \in [2,\infty)$ it holds that f' is increasing. This and the fact that (3.56) implies that $\lim_{x\to\infty} f'(x) = 0$ demonstrates that for all $x \in [2,\infty)$ it holds that $f'(x) \in (-\infty, 0]$. Hence, we obtain that for all $x \in [2,\infty)$ it holds that f is non-increasing. Combining this and the fact that $\lim_{x\to\infty} f(x) = 1$ assures that for all $x \in [2,\infty)$ it holds that $f(x) \in [1,\infty)$. This and (3.55) guarantee that for all $n \in \mathbb{N}$ it holds that

$$\ln(n!) = (n + \tfrac{1}{2})\ln(n) - n + 1 + \sum_{k=2}^{n}[1 - f(k)] \le (n + \tfrac{1}{2})\ln(n) - n + 1. \tag{3.58}$$

Furthermore, note that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} \sum_{k=2}^{n}[1 - f(k)] &= \sum_{k=2}^{n}\big[1 - (k - \tfrac{1}{2})\big(\ln(k) - \ln(k-1)\big)\big] \\ &= \mathbb{1}_{\mathbb{N}}(n-1)\big[(n-1) - (n - \tfrac{1}{2})\ln(n) + \ln((n-1)!)\big]. \end{aligned} \tag{3.59}$$

Combining this, the fact that for all $x \in [2,\infty)$ it holds that f is non-increasing, and (3.55) ensures that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} \sum_{k=2}^{n}[1 - f(k)] &= \mathbb{1}_{\mathbb{N}}(n-1)\big[(n-1) - (n - \tfrac{1}{2})\ln(n) + \ln((n-1)!)\big] \\ &= \mathbb{1}_{\mathbb{N}}(n-1)\big[\ln((n-1)!) - \big((n - \tfrac{1}{2})\ln(n-1) - (n-1) + 1\big) \\ &\quad + 1 - (n - \tfrac{1}{2})\ln(n) + (n - \tfrac{1}{2})\ln(n-1)\big] \\ &\ge \mathbb{1}_{\mathbb{N}}(n-1)[1 - f(n)] \ge \mathbb{1}_{\mathbb{N}}(n-1)[1 - \tfrac{3}{2}\ln(2)] \ge -\tfrac{1}{2}\ln(2). \end{aligned} \tag{3.60}$$

This and (3.55) show that for all $n \in \mathbb{N}$ it holds that

$$\ln(n!) = (n + \tfrac{1}{2})\ln(n) - n + 1 + \sum_{k=2}^{n}[1 - f(k)] \ge (n + \tfrac{1}{2})\ln(n) - n + 1 - \tfrac{1}{2}\ln(2). \tag{3.61}$$

Combining this and (3.58) proves that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} \frac{n^n}{\exp(n-1)} &\le 2^{-1/2}\frac{n^{(n+1/2)}}{\exp(n - 1/2)} = \exp\big((n + 1/2)\ln(n) - n + 1 - \ln(2^{1/2})\big) \le n! \\ &\le \exp((n + 1/2)\ln(n) - n + 1) = \frac{n^{(n+1/2)}}{\exp(n-1)} \le \frac{(n+1)^{(n+1)}}{\exp(n)}. \end{aligned} \tag{3.62}$$

The proof of Lemma 3.11 is thus complete. ☐

Lemma 3.12. *Let $\lfloor\cdot\rfloor\colon \mathbb{R} \to \mathbb{Z}$ satisfy for all $x \in \mathbb{R}$ that $\lfloor x \rfloor = \max\{n \in \mathbb{Z}\colon n \le x\}$. Then for all $m \in [1,\infty)$ it holds that*

$$\max_{n\in\mathbb{N}_0} \frac{m^{n/2}}{n!} \le \frac{m^{\lfloor m^{1/2}\rfloor/2}}{\lfloor m^{1/2}\rfloor!} < \frac{\exp(m^{1/2})}{(\lfloor m^{1/2}\rfloor)^{1/2}} \le \exp(m^{1/2}). \tag{3.63}$$

Proof of Lemma 3.12. Throughout this proof let $m \in [1,\infty)$, let $\lceil\cdot\rceil\colon \mathbb{R} \to \mathbb{Z}$ satisfy for all $x \in \mathbb{R}$ that $\lceil x \rceil = \min\{n \in \mathbb{Z}\colon x \le n\}$, and let $f\colon \mathbb{N}_0 \to \mathbb{R}$ satisfy for all $n \in \mathbb{N}_0$ that $f(n) = \ln(m^{n/2}) - \ln(n!)$. We claim that

$$\max_{n\in\mathbb{N}_0} f(n) = f(\lfloor m^{1/2}\rfloor). \tag{3.64}$$

Note that for all $n \in \mathbb{N}_0$ it holds that

$$f(n) = \ln(m^{n/2}) - \ln\bigl(n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1\bigr) = \tfrac{n}{2} \ln(m) - \textstyle\sum_{k=1}^{n} \ln(k). \tag{3.65}$$

This guarantees that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} f(n) - f(n-1) &= \bigl[\tfrac{n}{2} \ln(m) - \textstyle\sum_{k=1}^{n} \ln(k)\bigr] - \bigl[\tfrac{n-1}{2} \ln(m) - \textstyle\sum_{k=1}^{n-1} \ln(k)\bigr] \\ &= \tfrac{1}{2} \ln(m) - \ln(n) = \ln(m^{1/2}) - \ln(n). \end{aligned} \tag{3.66}$$

This and the fact that $(0,\infty) \ni x \mapsto \ln(x) \in \mathbb{R}$ is increasing show that for all $n \in \{1, 2, \ldots, \lfloor m^{1/2} \rfloor\}$ it holds that $f(n) - f(n-1) \geq 0$. Furthermore, note that (3.66) and the fact that $(0,\infty) \ni x \mapsto \ln(x) \in \mathbb{R}$ is increasing assure that for all $n \in \mathbb{N} \cap [\lceil m^{1/2} \rceil, \infty)$ it holds that $f(n) - f(n-1) \leq 0$. Combining this and the fact that for all $n \in \{1, 2, \ldots, \lfloor m^{1/2} \rfloor\}$ it holds that $f(n) - f(n-1) \geq 0$ demonstrates that

$$\max_{n \in \mathbb{N}_0} f(n) = \max\{f(\lfloor m^{1/2} \rfloor), f(\lceil m^{1/2} \rceil)\}. \tag{3.67}$$

Next observe that (3.65), the fact that $(0,\infty) \ni x \mapsto \ln(x) \in \mathbb{R}$ is increasing, and the fact that for all $x \in \mathbb{R}$ it holds that $\lfloor x \rfloor \leq \lceil x \rceil$ guarantee that

$$\begin{aligned} &f(\lceil m^{1/2} \rceil) - f(\lfloor m^{1/2} \rfloor) \\ &= \bigl(\lceil m^{1/2} \rceil \ln(m^{1/2}) - \textstyle\sum_{k=1}^{\lceil m^{1/2} \rceil} \ln(k)\bigr) - \bigl(\lfloor m^{1/2} \rfloor \ln(m^{1/2}) - \textstyle\sum_{k=1}^{\lfloor m^{1/2} \rfloor} \ln(k)\bigr) \\ &= \bigl(\lceil m^{1/2} \rceil - \lfloor m^{1/2} \rfloor\bigr) \ln(m^{1/2}) - \bigl(\textstyle\sum_{k=1}^{\lceil m^{1/2} \rceil} \ln(k) - \textstyle\sum_{k=1}^{\lfloor m^{1/2} \rfloor} \ln(k)\bigr) \\ &= \mathbb{1}_{\mathbb{R} \setminus \mathbb{N}}(m^{1/2}) \ln(m^{1/2}) - \textstyle\sum_{k=\lfloor m^{1/2} \rfloor + 1}^{\lceil m^{1/2} \rceil} \ln(k) = \mathbb{1}_{\mathbb{R} \setminus \mathbb{N}}(m^{1/2}) \bigl[\ln(m^{1/2}) - \ln(\lceil m^{1/2} \rceil)\bigr] \leq 0. \end{aligned} \tag{3.68}$$

Combining this and (3.67) establishes (3.64). In addition, observe that Lemma 3.11, the fact that for all $x \in \mathbb{R}$ it holds that $\lfloor x \rfloor \leq x$, and the fact that $\ln(6) < 2$ ensure that

$$\begin{aligned} f(\lfloor m^{1/2} \rfloor) &= \ln(m^{\lfloor m^{1/2} \rfloor / 2}) - \ln(\lfloor m^{1/2} \rfloor!) \\ &\leq \ln(m^{\lfloor m^{1/2} \rfloor / 2}) - \bigl(\lfloor m^{1/2} \rfloor + \tfrac{1}{2}\bigr) \ln(\lfloor m^{1/2} \rfloor) + \lfloor m^{1/2} \rfloor - 1 + \tfrac{1}{2} \ln(2) \\ &= \bigl[\ln(m^{\lfloor m^{1/2} \rfloor / 2}) - \lfloor m^{1/2} \rfloor \ln(\lfloor m^{1/2} \rfloor)\bigr] - \tfrac{1}{2} \ln(\lfloor m^{1/2} \rfloor) + \lfloor m^{1/2} \rfloor - 1 + \tfrac{1}{2} \ln(2) \\ &\leq \tfrac{1}{2} \ln(3) - \tfrac{1}{2} \ln(\lfloor m^{1/2} \rfloor) + \lfloor m^{1/2} \rfloor - 1 + \tfrac{1}{2} \ln(2) \\ &< \lfloor m^{1/2} \rfloor - \tfrac{1}{2} \ln(\lfloor m^{1/2} \rfloor) \leq m^{1/2} - \tfrac{1}{2} \ln(\lfloor m^{1/2} \rfloor). \end{aligned} \tag{3.69}$$

Combining this, (3.64), and the fact that $\mathbb{R} \ni x \mapsto \exp(x) \in (0,\infty)$ is monotone yields that

$$\begin{aligned} \max_{n \in \mathbb{N}_0} \exp(f(n)) &= \max_{n \in \mathbb{N}_0} \frac{m^{n/2}}{n!} \leq \frac{m^{\lfloor m^{1/2} \rfloor / 2}}{\lfloor m^{1/2} \rfloor!} < \exp\bigl(m^{1/2} - \tfrac{1}{2} \ln(\lfloor m^{1/2} \rfloor)\bigr) \\ &= \frac{\exp(m^{1/2})}{(\lfloor m^{1/2} \rfloor)^{1/2}} \leq \exp(m^{1/2}). \end{aligned} \tag{3.70}$$

The proof of Lemma 3.12 is thus complete. □

Lemma 3.13. *Let $M, N \in \mathbb{N}$, $T \in (0,\infty)$, $\tau \in [0,T]$, $a, b \in [0,\infty)$, $p \in [1,\infty)$, let $\lfloor \cdot \rfloor \colon \mathbb{R} \to \mathbb{Z}$ satisfy for all $x \in \mathbb{R}$ that $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \colon n \leq x\}$, let $f_n \colon [\tau,T] \to [0,\infty]$, $n \in \mathbb{N}_0$, be measurable, assume $\sup_{s \in [\tau,T]} |f_0(s)| < \infty$, and assume for all $n \in \{1, 2, \ldots, N\}$, $t \in [\tau,T]$ that*

$$|f_n(t)| \leq \frac{a}{M^{n/2}} + \sum_{i=0}^{n-1} \left[\frac{b}{M^{(n-i-1)/2}} \left[\int_t^T |f_i(s)|^p \, ds \right]^{1/p} \right]. \tag{3.71}$$

Then

$$f_N(\tau) \le \left[a + b(T-\tau)^{1/p}\left[\sup_{s\in[\tau,T]}|f_0(s)|\right]\right] \frac{\left(1+b(T-\tau)^{1/p}\right)^{N-1}}{M^{(N-\lfloor M^{p/2}\rfloor)/2}(\lfloor M^{p/2}\rfloor!)^{1/p}} \le \left[a + b(T-\tau)^{1/p}\left[\sup_{s\in[\tau,T]}|f_0(s)|\right]\right] \frac{\left(1+b(T-\tau)^{1/p}\right)^{N-1}}{M^{N/2}\exp\left(-M^{p/2}/p\right)}. \tag{3.72}$$

Proof of Lemma 3.13. Note that Hutzenthaler et al. [28, Lemma 3.10] (applied with $c \curvearrowleft M^{-1/2}$, $\alpha \curvearrowleft \tau$, $\beta \curvearrowleft T$ in the notation of Hutzenthaler et al. [28, Lemma 3.10]) assures that

$$f_N(\tau) \le \left[a + b(T-\tau)^{1/p}\left[\sup_{s\in[\tau,T]}|f_0(s)|\right]\right]\left[\sup_{k\in\mathbb{N}_0}\tfrac{M^{k/2}}{(k!)^{1/p}}\right]\left(1+b(T-\tau)^{1/p}\right)^{N-1}. \tag{3.73}$$

This, the fact that $a, b \in [0,\infty)$, and Lemma 3.12 (applied with $M \curvearrowleft M^p$ in the notation of Lemma 3.12) prove that

$$f_N(\tau) \le \left[a + b(T-\tau)^{1/p}\left[\sup_{s\in[\tau,T]}|f_0(s)|\right]\right] \frac{\left(1+b(T-\tau)^{1/p}\right)^{N-1}}{M^{(N-\lfloor M^{p/2}\rfloor)/2}(\lfloor M^{p/2}\rfloor!)^{1/p}} \le \left[a + b(T-\tau)^{1/p}\left[\sup_{s\in[\tau,T]}|f_0(s)|\right]\right] \frac{\left(1+b(T-\tau)^{1/p}\right)^{N-1}}{M^{N/2}\exp\left(-M^{p/2}/p\right)}. \tag{3.74}$$

The proof of Lemma 3.13 is thus complete. □

3.7 Non-recursive error bounds for MLP approximations

Lemma 3.14. *Assume Setting 3.2 and let $\lfloor\cdot\rfloor\colon \mathbb{R}\to\mathbb{Z}$ satisfy for all $x\in\mathbb{R}$ that $\lfloor x\rfloor = \max\{n\in\mathbb{Z}\colon n\le x\}$. Then it holds for all $n\in\mathbb{N}_0$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that*

$$\begin{aligned}&\left(\mathbb{E}\left[|U^0_n(t,x+W^0_t)-u(t,x+W^0_t)|^{\mathfrak{p}}\right]\right)^{1/\mathfrak{p}}\\ &\le \frac{\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)(1+2LT)^n}{\mathfrak{m}^{(n-\lfloor \mathfrak{m}^{\mathfrak{p}/2}\rfloor)/2}(\lfloor \mathfrak{m}^{\mathfrak{p}/2}\rfloor!)^{1/\mathfrak{p}}}\left[\sup_{s\in[0,T]}\left(\mathbb{E}\left[\left(1+\|x+W^0_s\|^p\right)^{\mathfrak{p}}\right]\right)^{1/\mathfrak{p}}\right]\\ &\le \frac{\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)(1+2LT)^n}{\mathfrak{m}^{n/2}\exp\left(-\mathfrak{m}^{\mathfrak{p}/2}/\mathfrak{p}\right)}\left[\sup_{s\in[0,T]}\left(\mathbb{E}\left[\left(1+\|x+W^0_s\|^p\right)^{\mathfrak{p}}\right]\right)^{1/\mathfrak{p}}\right].\end{aligned} \tag{3.75}$$

Proof of Lemma 3.14. Throughout this proof assume without loss of generality that $T\in(0,\infty)$, let $t\in[0,T]$, $b = 2LT^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}$, let $a\colon\mathbb{R}^d\to\mathbb{R}$ satisfy

$$a(x) = \mathfrak{m}\exp(LT)\left[\left(\mathbb{E}\left[|g(x+W^0_T)|^{\mathfrak{p}}\right]\right)^{1/\mathfrak{p}} + T^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\int_0^T \mathbb{E}\left[|f(s,x+W^0_s,0)|^{\mathfrak{p}}\right]ds\right)^{1/\mathfrak{p}}\right], \tag{3.76}$$

and let $\mathfrak{f}_{n,k}\colon[t,T]\times\mathbb{R}^d\to[0,\infty]$, $n\in\mathbb{N}_0$, $k\in\{0,1,\dots,n\}$, satisfy for all $n\in\mathbb{N}_0$, $k\in\{0,1,\dots,n\}$, $r\in[t,T]$, $x\in\mathbb{R}^d$ that

$$\mathfrak{f}_{n,k}(r,x) = \left(\mathbb{E}\left[|U^0_k(r,x+W^0_r)-u(r,x+W^0_r)|^{\mathfrak{p}}\right]\right)^{1/\mathfrak{p}}. \tag{3.77}$$

Note that (3.77) ensures that for all $n\in\mathbb{N}_0$, $k\in\{0,1,\dots,n\}$ it holds that $\mathfrak{f}_{n,k}$ is measurable. In addition, observe that (3.76), (3.77), and Lemma 3.10 assure that for all $n\in\mathbb{N}_0$, $k\in\{0,1,\dots,n\}$, $r\in[t,T]$, $x\in\mathbb{R}^d$ it holds that

$$|\mathfrak{f}_{n,k}(r,x)| = \left(\mathbb{E}\left[|U^0_k(r,x+W^0_r)-u(r,x+W^0_r)|^{\mathfrak{p}}\right]\right)^{1/\mathfrak{p}}$$

$$
\begin{aligned}
&\leq \frac{\mathfrak{m}\exp(L(T-r))}{m^{n/2}}\left[\left(\mathbb{E}\big[|g(x+W^0_T)|^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}} + (T-r)^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\int_r^T \mathbb{E}\big[|f(s,x+W^0_s,0)|^{\mathfrak{p}}\big]\,ds\right)^{1/\mathfrak{p}}\right]\\
&\quad+\sum_{i=0}^{k-1}\frac{L(T-r)^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i)/2}}\left[\left(\mathbb{1}_{(0,n)}(i)+m^{1/2}\right)\left(\int_r^T \mathbb{E}\big[|(U^0_{i-1}-u)(s,x+W^0_s)|^{\mathfrak{p}}\big]\,ds\right)^{1/\mathfrak{p}}\right]\\
&\leq \frac{\mathfrak{m}\exp(LT)}{m^{n/2}}\left[\left(\mathbb{E}\big[|g(x+W^0_T)|^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}} + T^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\int_0^T \mathbb{E}\big[|f(s,x+W^0_s,0)|^{\mathfrak{p}}\big]\,ds\right)^{1/\mathfrak{p}}\right]\\
&\quad+\sum_{i=0}^{k-1}\frac{2LT^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}}{m^{(n-i-1)/2}}\left(\int_r^T \mathbb{E}\big[|(U^0_{i-1}-u)(s,x+W^0_s)|^{\mathfrak{p}}\big]\,ds\right)^{1/\mathfrak{p}}\\
&= \frac{a(x)}{m^{n/2}}+\sum_{i=0}^{k-1}\left[\frac{b}{m^{(n-i-1)/2}}\left[\int_r^T |\mathfrak{f}_{n,i}(s,x)|^p\,ds\right]^{1/p}\right].
\end{aligned}
\tag{3.78}
$$

Next note that (3.5), (3.76), and Corollary 2.5 assure that for all $n\in\mathbb{N}_0$, $r\in[t,T]$, $x\in\mathbb{R}^d$ it holds that

$$
\begin{aligned}
|\mathfrak{f}_{n,0}(r,x)| &= \left(\mathbb{E}\big[|U^0_0(r,x+W^0_r)-u(r,x+W^0_r)|^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}} = \left(\mathbb{E}\big[|u(r,x+W^0_r)|^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}}\\
&\leq \mathfrak{L}(T+1)\exp(LT)\left[\sup_{s\in[0,T]}\left(\mathbb{E}\big[(1+\|x+W_s\|^p)^q\big]\right)^{1/q}\right]<\infty.
\end{aligned}
\tag{3.79}
$$

Combining this, (3.76), (3.77), (3.78), and Lemma 3.13 (applied for every $n\in\mathbb{N}$, $x\in\mathbb{R}^d$ with $a \curvearrowleft a(x)$, $b\curvearrowleft b$, $N\curvearrowleft n$, $\tau\curvearrowleft t$, $T\curvearrowleft T$, $(f_k)_{k\in\{0,1,\dots,n\}}\curvearrowleft ([t,T]\ni r\mapsto \mathfrak{f}_{n,k}(r,x)\in[0,\infty])_{k\in\{0,1,\dots,n\}}$ in the notation of Lemma 3.13) guarantees that for all $n\in\mathbb{N}$, $x\in\mathbb{R}^d$ it holds that

$$
\begin{aligned}
&\left(\mathbb{E}\big[|U^0_n(t,x+W^0_t)-u(t,x+W^0_t)|^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}} = \mathfrak{f}_{n,n}(t,x)\\
&\leq \left[a(x)+b(T-t)^{1/\mathfrak{p}}\left[\sup_{s\in[t,T]}|\mathfrak{f}_{n,0}(s,x)|\right]\right]\frac{\left(1+b(T-t)^{1/\mathfrak{p}}\right)^{n-1}}{m^{(n-\lfloor m^{\mathfrak{p}/2}\rfloor)/2}\left(\lfloor m^{\mathfrak{p}/2}\rfloor!\right)^{1/\mathfrak{p}}}\\
&\leq \left[a(x)+b(T-t)^{1/\mathfrak{p}}\left[\sup_{s\in[t,T]}|\mathfrak{f}_{n,0}(s,x)|\right]\right]\frac{\left(1+b(T-t)^{1/\mathfrak{p}}\right)^{n-1}}{m^{n/2}\exp\left(-m^{\mathfrak{p}/2}/\mathfrak{p}\right)}.
\end{aligned}
\tag{3.80}
$$

In addition, observe that (3.2) demonstrates that for all $x\in\mathbb{R}^d$ it holds that

$$
\begin{aligned}
&\left(\mathbb{E}\big[|g(x+W^0_T)|^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}} + T^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\int_0^T \mathbb{E}\big[|f(s,x+W^0_s,0)|^{\mathfrak{p}}\big]\,ds\right)^{1/\mathfrak{p}}\\
&\leq \mathfrak{L}\left(\mathbb{E}\big[(1+\|x+W^0_T\|^p)^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}} + T^{(\mathfrak{p}-1)/\mathfrak{p}}\left(\mathfrak{L}T\sup_{s\in[0,T]}\mathbb{E}\big[(1+\|x+W^0_s\|^p)^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}}\\
&\leq \mathfrak{L}(T+1)\left[\sup_{s\in[0,T]}\left(\mathbb{E}\big[(1+\|x+W^0_s\|^p)^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}}\right].
\end{aligned}
\tag{3.81}
$$

This, the fact that $b=2LT^{(\mathfrak{p}-1)/\mathfrak{p}}\mathfrak{m}$, (3.76), and (3.79) show that for all $n\in\mathbb{N}$, $x\in\mathbb{R}^d$ it holds that

$$
\begin{aligned}
&a(x)+b(T-t)^{1/\mathfrak{p}}\left[\sup_{s\in[t,T]}|\mathfrak{f}_{n,0}(s,x)|\right]\\
&\leq \left[1+2LT^{(\mathfrak{p}-1)/\mathfrak{p}}(T-t)^{1/\mathfrak{p}}\right]\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)\left[\sup_{s\in[0,T]}\left(\mathbb{E}\big[(1+\|x+W^0_s\|^p)^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}}\right]\\
&\leq [1+2LT]\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)\left[\sup_{s\in[0,T]}\left(\mathbb{E}\big[(1+\|x+W^0_s\|^p)^{\mathfrak{p}}\big]\right)^{1/\mathfrak{p}}\right].
\end{aligned}
\tag{3.82}
$$

Combining this, (3.80), and the fact that for all $n \in \mathbb{N}$ it holds that

$$\big(1 + b(T-t)^{1/\mathfrak{p}}\big)^{n-1} = \big(1 + 2LT^{(\mathfrak{p}-1)/\mathfrak{p}}(T-t)^{1/\mathfrak{p}}\big)^{n-1} \leq (1 + 2LT)^{n-1} \tag{3.83}$$

proves that for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &\big(\mathbb{E}\big[|U_n^0(t, x + W_t^0) - u(t, x + W_t^0)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} \\ &\leq \frac{\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)(1+2LT)^n}{m^{(n-\lfloor m^{\mathfrak{p}/2}\rfloor)/2}(\lfloor m^{\mathfrak{p}/2}\rfloor!)^{1/\mathfrak{p}}}\left[\sup_{s\in[0,T]}\big(\mathbb{E}\big[(1 + \|x + W_s^0\|^p)^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}\right] \\ &\leq \frac{\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)(1+2LT)^n}{m^{n/2}\exp\big(-m^{\mathfrak{p}/2}/\mathfrak{p}\big)}\left[\sup_{s\in[0,T]}\big(\mathbb{E}\big[(1 + \|x + W_s^0\|^p)^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}\right]. \end{aligned} \tag{3.84}$$

Combining this and (3.79) establishes (3.75). The proof of Lemma 3.14 is thus complete. ☐

Corollary 3.15. *Assume Setting 3.2. Then it holds for all $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that*

$$\begin{aligned} &\Big(\mathbb{E}\Big[\big|U_n^0(t,x) - u(t,x)\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}} \\ &\leq \frac{\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)(1+2LT)^n}{m^{n/2}\exp\big(-m^{\mathfrak{p}/2}/\mathfrak{p}\big)}\left[\sup_{s\in[0,T]}\Big(\mathbb{E}\Big[\big(1 + \|x + W_s^0\|^p\big)^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}}\right]. \end{aligned} \tag{3.85}$$

Proof of Corollary 3.15. Throughout this proof let $V_\mathfrak{t}\colon [0, T-\mathfrak{t}] \times \mathbb{R}^d \to \mathbb{R}$, $\mathfrak{t} \in [0,T]$, satisfy for all $\mathfrak{t} \in [0,T]$, $t \in [0, T-\mathfrak{t}]$, $x \in \mathbb{R}^d$ that

$$V_\mathfrak{t}(t,x) = u(t+\mathfrak{t}, x), \tag{3.86}$$

let $G_\mathfrak{t}\colon C([0,T-\mathfrak{t}]\times\mathbb{R}^d,\mathbb{R}) \to C([0,T-\mathfrak{t}]\times\mathbb{R}^d,\mathbb{R})$, $\mathfrak{t} \in [0,T]$, satisfy for all $\mathfrak{t} \in [0,T]$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$, $v \in C([0,T-\mathfrak{t}]\times\mathbb{R}^d,\mathbb{R})$ that

$$(G_\mathfrak{t}(v))(t,x) = (F(v))(t+\mathfrak{t},x), \tag{3.87}$$

let $\mathcal{R}^{\mathfrak{t},\theta}\colon [0,T-\mathfrak{t}]\times\Omega \to [0,T-\mathfrak{t}]$, $\mathfrak{t} \in [0,T]$, $\theta \in \Theta$, satisfy for all $\mathfrak{t} \in [0,T]$, $t \in [0,T-\mathfrak{t}]$, $\theta \in \Theta$ that

$$\mathcal{R}_t^{\theta,\mathfrak{t}} = t + (T - (t+\mathfrak{t}))\mathfrak{u}^\theta, \tag{3.88}$$

and let $\mathfrak{V}_n^{\theta,\mathfrak{t}}\colon [0,T-\mathfrak{t}]\times\mathbb{R}^d\times\Omega \to \mathbb{R}$, $\mathfrak{t} \in [0,T]$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, satisfy for all $\mathfrak{t} \in [0,T]$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$ that

$$\mathfrak{V}_n^{\theta,\mathfrak{t}}(t,x) = U_n^\theta(t+\mathfrak{t},x). \tag{3.89}$$

Observe that (3.4), (3.86), (3.87), and the fact that W^0 has independent increments ensure that for all $\mathfrak{t} \in [0,T]$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} V_\mathfrak{t}(t,x) = u(t+\mathfrak{t},x) &= \mathbb{E}\big[g(x + W_{T-(t+\mathfrak{t})}^0)\big] + \int_{(t+\mathfrak{t})}^T \mathbb{E}\big[(F(u))(s, x + W_{s-(t+\mathfrak{t})}^0)\big]\,ds \\ &= \mathbb{E}\big[g(x + W_{(T-\mathfrak{t})-t}^0)\big] + \int_t^{(T-\mathfrak{t})} \mathbb{E}\big[(F(u))(s+\mathfrak{t}, x + W_{s-t}^0)\big]\,ds \\ &= \mathbb{E}\big[g(x + W_{(T-\mathfrak{t})-t}^0)\big] + \int_t^{(T-\mathfrak{t})} \mathbb{E}\big[(G_\mathfrak{t}(V_\mathfrak{t}))(s, x + W_{s-t}^0)\big]\,ds. \end{aligned} \tag{3.90}$$

Combining this, (3.86), (3.87), and the hypothesis that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $\mathbb{E}[|g(x + W_{T-t}^0)| + \int_t^T |F(u))(s, x + W_{s-t}^0)|\,ds] < \infty$ implies that for all $\mathfrak{t} \in [0,T]$, $t \in [0,T-\mathfrak{t}]$,

$x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathbb{E}\bigg[\big|g(x+W^0_{(T-\mathfrak{t})-t})\big| + \int_t^{(T-\mathfrak{t})} \big|(G_\mathfrak{t}(V_\mathfrak{t}))(s,x+W^0_{s-t})\big|\,ds\bigg] \\ &= \mathbb{E}\bigg[\big|g(x+W^0_{T-(t+\mathfrak{t})})\big| + \int_{(t+\mathfrak{t})}^T \big|(F(u))(s,x+W^0_{s-(t+\mathfrak{t})})\big|\,ds\bigg] < \infty.\end{aligned} \tag{3.91}$$

Next note that (3.2), (3.3), and (3.87) demonstrate that for all $\mathfrak{t} \in [0,T]$, $t \in [0, T-\mathfrak{t}]$, $x \in \mathbb{R}^d$, $v, w \in C([0,T-\mathfrak{t}]\times\mathbb{R}^d,\mathbb{R})$ it holds that

$$\begin{aligned}&|(G_\mathfrak{t}(v))(t,x) - (G_\mathfrak{t}(w))(t,x)| = |(F(v))(t+\mathfrak{t},x) - (F(w))(t+\mathfrak{t},x)| \\ &= |f(t+\mathfrak{t},x,v(t+\mathfrak{t},x)) - f(t+\mathfrak{t},x,w(t+\mathfrak{t},x))| \leq L|v(t+\mathfrak{t},x) - w(t+\mathfrak{t},x)|.\end{aligned} \tag{3.92}$$

In addition, observe that (3.2), (3.3), and (3.87) show that for all $\mathfrak{t} \in [0,T]$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$ it holds that

$$|(G_\mathfrak{t}(0))(t,x)| = |f(t+\mathfrak{t},x,0)| \leq \mathfrak{L}(1+\|x\|^p). \tag{3.93}$$

Moreover, note that (3.86), (3.90), and the hypothesis that $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ assure that for all $\mathfrak{t} \in [0,T]$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$ it holds that $V_\mathfrak{t} \in C([0,T-\mathfrak{t}]\times\mathbb{R}^d,\mathbb{R})$. Furthermore, observe that (3.88), the hypothesis that $(\mathfrak{u}^\theta)_{\theta\in\Theta}$ are i.i.d. random variables, and the hypothesis that $(W^\theta)_{\theta\in\Theta}$ and $(\mathcal{U}^\theta)_{\theta\in\Theta}$ are independent ensure that for all $\mathfrak{t} \in [0,T]$ it holds that $(W^\theta)_{\theta\in\Theta}$ and $(\mathcal{R}^{\theta,\mathfrak{t}})_{\theta\in\Theta}$ are independent on $[0,T-\mathfrak{t}]$. Next note that (3.89) implies that for all $\mathfrak{t} \in [0,T]$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathfrak{V}^{\theta,\mathfrak{t}}_n(t,x) = U^\theta_n(t+\mathfrak{t},x) = \tfrac{\mathbb{1}_\mathbb{N}(n)}{m^n}\Bigg[\sum_{k=1}^{m^n} g(x+W^{(\theta,0,-k)}_{T-(t+\mathfrak{t})})\Bigg] \\ &+ \sum_{i=0}^{n-1} \tfrac{(T-(t+\mathfrak{t}))}{m^{n-i}}\Bigg[\sum_{k=1}^{m^{n-i}} \big[(F(U^{(\theta,i,k)}_i) - \mathbb{1}_\mathbb{N}(i)F(U^{(\theta,-i,k)}_{i-1}))(\mathcal{U}^{(\theta,i,k)}_{(t+\mathfrak{t})}, x+W^{(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_{(t+\mathfrak{t})}-(t+\mathfrak{t})})\big]\Bigg].\end{aligned} \tag{3.94}$$

Combining this, the fact that for all $t \in [0,T]$, $\theta \in \Theta$ that $\mathcal{U}_t = t + (T-t)\mathfrak{u}^\theta$, (3.87), (3.88), and (3.89) shows that for all $\mathfrak{t} \in [0,T]$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&\mathfrak{V}^{\theta,\mathfrak{t}}_n(t,x) = \tfrac{\mathbb{1}_\mathbb{N}(n)}{m^n}\Bigg[\sum_{k=1}^{m^n} g(x+W^{(\theta,0,-k)}_{(T-\mathfrak{t})-t})\Bigg] \\ &+ \sum_{i=0}^{n-1} \tfrac{(T-\mathfrak{t})-t)}{m^{n-i}}\Bigg[\textstyle\sum_{k=1}^{m^{n-i}} \big[(F(U^{(\theta,i,k)}_i) - \mathbb{1}_\mathbb{N}(i)F(U^{(\theta,-i,k)}_{i-1}))(\mathfrak{t}+\mathcal{R}^{(\theta,i,k),\mathfrak{t}}_t, x+W^{(\theta,i,k)}_{\mathcal{R}^{(\theta,i,k),\mathfrak{t}}_t-t})\big]\Bigg] \\ &= \tfrac{\mathbb{1}_\mathbb{N}(n)}{m^n}\Bigg[\sum_{k=1}^{m^n} g(x+W^{(\theta,0,-k)}_{(T-\mathfrak{t})-t})\Bigg] \\ &+ \sum_{i=0}^{n-1} \tfrac{(T-\mathfrak{t})-t)}{m^{n-i}}\Bigg[\textstyle\sum_{k=1}^{m^{n-i}} \big[(G_\mathfrak{t}(\mathfrak{V}^{(\theta,i,k),\mathfrak{t}}_i) - \mathbb{1}_\mathbb{N}(i)G_\mathfrak{t}(\mathfrak{V}^{(\theta,-i,k),\mathfrak{t}}_{i-1}))(\mathcal{R}^{(\theta,i,k),\mathfrak{t}}_t, x+W^{(\theta,i,k)}_{\mathcal{R}^{(\theta,i,k),\mathfrak{t}}_t-t})\big]\Bigg].\end{aligned} \tag{3.95}$$

Combining this, (3.90), (3.91), (3.92), (3.93), the fact that $1+m^{-1/2} \leq 2$, the fact that for all $\mathfrak{t} \in [0,T]$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$ it holds that $V_\mathfrak{t} \in C([0,T-\mathfrak{t}]\times\mathbb{R}^d,\mathbb{R})$, and Lemma 3.14 (applied for every $\mathfrak{t} \in [0,T]$ with $L \curvearrowleft L$, $\mathfrak{L} \curvearrowleft \mathfrak{L}$, $p \curvearrowleft p$, $\mathfrak{p} \curvearrowleft \mathfrak{p}$, $T \curvearrowleft (T-\mathfrak{t})$, $g \curvearrowleft g$, $F \curvearrowleft G_\mathfrak{t}$, $(\mathcal{U}^\theta)_{\theta\in\Theta} \curvearrowleft (\mathcal{R}^{\theta,\mathfrak{t}})_{\theta\in\Theta}$, $u \curvearrowleft V_\mathfrak{t}$, $(U^\theta_n)_{(n,\theta)\in\mathbb{N}_0\times\Theta} \curvearrowleft (\mathfrak{V}^{\theta,\mathfrak{t}}_n)_{(n,\theta)\in\mathbb{N}_0\times\Theta}$ in the notation of Lemma 3.14) demonstrates that for all $\mathfrak{t} \in [0,T]$, $t \in [0,T-\mathfrak{t}]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}_0$ it holds that

$$\begin{aligned}&\big(\mathbb{E}\big[|\mathfrak{V}^{0,\mathfrak{t}}_n(t,x+W^0_t) - V_\mathfrak{t}(t,x+W^0_t)|^\mathfrak{p}\big]\big)^{1/\mathfrak{p}} \\ &\leq \frac{\mathfrak{m}\mathfrak{L}((T-\mathfrak{t})+1)\exp(L(T-\mathfrak{t}))(1+2L(T-\mathfrak{t}))^n}{m^{n/2}\exp\big(-m^{\mathfrak{p}/2}/\mathfrak{p}\big)}\Bigg[\sup_{s\in[0,T-\mathfrak{t}]}\big(\mathbb{E}\big[(1+\|x+W^0_s\|^p)^\mathfrak{p}\big]\big)^{1/\mathfrak{p}}\Bigg]\end{aligned} \tag{3.96}$$

$$\le \frac{\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)(1+2LT)^n}{m^{n/2}\exp\bigl(-m^{\mathfrak{p}/2}/\mathfrak{p}\bigr)}\Biggl[\sup_{s\in[0,T]}\bigl(\mathbb{E}\bigl[(1+\|x+W_s^0\|^p)^{\mathfrak{p}}\bigr]\bigr)^{1/\mathfrak{p}}\Biggr].$$

This and (3.89) prove that for all $\mathfrak{t}\in[0,T]$, $x\in\mathbb{R}^d$, $n\in\mathbb{N}_0$ it holds that

$$\begin{aligned}&\bigl(\mathbb{E}\bigl[|U_n^0(\mathfrak{t},x)-u(\mathfrak{t},x)|^{\mathfrak{p}}\bigr]\bigr)^{1/\mathfrak{p}}=\bigl(\mathbb{E}\bigl[|\mathfrak{V}_n^{0,\mathfrak{t}}(0,x+W_0^0)-V_{\mathfrak{t}}(0,x+W_0^0)|^{\mathfrak{p}}\bigr]\bigr)^{1/\mathfrak{p}}\\&\le \frac{\mathfrak{m}\mathfrak{L}(T+1)\exp(LT)(1+2LT)^n}{m^{n/2}\exp\bigl(-m^{\mathfrak{p}/2}/\mathfrak{p}\bigr)}\Biggl[\sup_{s\in[0,T]}\bigl(\mathbb{E}\bigl[(1+\|x+W_s^0\|^p)^{\mathfrak{p}}\bigr]\bigr)^{1/\mathfrak{p}}\Biggr].\end{aligned}\tag{3.97}$$

The proof of Corollary 3.15 is thus complete. □

4 Computational complexity analysis for MLP approximations

In this section we use the results from Section 3 to provide the complexity analysis for MLP approximations of solutions to stochastic fixed-point equations and semilinear PDEs. The main result of this section is Theorem 4.6 in Subsection 4.4 below. The proof of Theorem 4.6 employs Proposition 4.4 and the elementary auxiliary result in Lemma 4.5. The proof of Proposition 4.4, in turn, is based on Corollary 3.15 and the elementary estimate for full-history recursions in Corollary 4.3. Our proof of Corollary 4.3 employs the elementary result for full-history recursions in Lemma 4.2. Our proof of Lemma 4.2, in turn, is based on the elementary result for two-step recursions in Lemma 4.1. Lemma 4.1 is a special case of Hutzenthaler et al. [33, Lemma 2.1]. Only for completeness we include in this section the detailed proof of Lemma 4.1.

4.1 Elementary estimates for two-step recursions

Lemma 4.1. *Let $\beta_1,\beta_2,b_1,b_2,\alpha_0,\alpha_1,\alpha_2,\ldots\in\mathbb{C}$ and let $x_k\in\mathbb{C}$, $k\in\mathbb{N}_0$, satisfy for all $k\in\mathbb{N}_0$, $j\in\{1,2\}$ that $x_k=\alpha_k+\mathbb{1}_{[1,\infty)}(k)\,\beta_1\,x_{\max\{k-1,0\}}+\mathbb{1}_{[2,\infty)}(k)\,\beta_2\,x_{\max\{k-2,0\}}$, $(\beta_1)^2\neq-4\beta_2$, and $b_j=\frac{1}{2}\bigl(\beta_1-(-1)^j\sqrt{(\beta_1)^2+4\beta_2}\bigr)$. Then it holds for all $k\in\mathbb{N}_0$ that $b_1-b_2=\sqrt{(\beta_1)^2+4\beta_2}\neq0$ and*

$$\begin{aligned}x_k&=\frac{1}{(b_1-b_2)}\sum_{l=0}^{k}\alpha_l\bigl([b_1]^{k+1-l}-[b_2]^{k+1-l}\bigr)\\&=\sum_{l=0}^{k}\frac{\alpha_l\bigl(\bigl(\beta_1+\sqrt{(\beta_1)^2+4\beta_2}\bigr)^{k+1-l}-\bigl[\beta_1-\sqrt{(\beta_1)^2+4\beta_2}\bigr]^{k+1-l}\bigr)}{2^{(k+1-l)}\sqrt{(\beta_1)^2+4\beta_2}}.\end{aligned}\tag{4.1}$$

Proof of Lemma 4.1. Throughout this proof let $x_{-1},x_{-2}\in\mathbb{C}$ satisfy that $x_{-1}=x_{-2}=0$ and let $y_k\in\mathbb{C}$, $k\in\mathbb{N}_0$, satisfy for all $k\in\mathbb{N}_0$ that

$$y_k=\frac{1}{(b_2-b_1)}\sum_{l=0}^{k}\alpha_l\bigl([b_2]^{k+1-l}-[b_1]^{k+1-l}\bigr).\tag{4.2}$$

Note that (4.2) and the fact that $x_0=\alpha_0$ ensure that

$$y_0=\frac{1}{(b_2-b_1)}\sum_{l=0}^{0}\alpha_l\bigl([b_2]^{1-l}-[b_1]^{1-l}\bigr)=\tfrac{1}{(b_2-b_1)}\cdot\alpha_0\cdot\bigl(b_2-b_1\bigr)=\alpha_0=x_0.\tag{4.3}$$

This, (4.2), the fact that for all $k\in\mathbb{N}_0$ it holds that $x_k=\alpha_k+\beta_1x_{k-1}+\beta_2x_{k-2}$, the fact that $x_{-1}=x_{-2}=0$, and the fact that $b_1+b_2=\beta_1$ prove that

$$\begin{aligned}y_1&=\frac{1}{(b_2-b_1)}\sum_{l=0}^{1}\alpha_l\bigl([b_2]^{2-l}-[b_1]^{2-l}\bigr)=\tfrac{1}{(b_2-b_1)}\bigl[\alpha_0\bigl([b_2]^2-[b_1]^2\bigr)+\alpha_1(b_2-b_1)\bigr]\\&=\alpha_0(b_1+b_2)+\alpha_1=\alpha_0\beta_1+\alpha_1=x_0\beta_1+\alpha_1=\alpha_1+\beta_1x_0+\beta_2x_{-1}=x_1.\end{aligned}\tag{4.4}$$

Next observe that the quadratic formula implies that for all $j \in \{1,2\}$ it holds that $(b_j)^2 = \beta_1 b_j + \beta_2$. This and (4.2) assure that for all $k \in \mathbb{N}_0$ it holds that

$$\begin{aligned} y_{k+2} &= \frac{1}{(b_2 - b_1)} \sum_{l=0}^{k+2} \alpha_l \big([b_2]^{k+3-l} - [b_1]^{k+3-l}\big) = \tfrac{\alpha_{k+2}(b_2-b_1)}{(b_2-b_1)} + \sum_{l=0}^{k+1} \tfrac{\alpha_l}{(b_2-b_1)} \big([b_2]^{k+3-l} - [b_1]^{k+3-l}\big) \\ &= \alpha_{k+2} + \sum_{l=0}^{k+1} \tfrac{\alpha_l}{(b_2-b_1)} \big([b_2]^{k+1-l}[\beta_1 b_2 + \beta_2] - [b_1]^{k+1-l}[\beta_1 b_1 + \beta_2]\big) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} y_{k+2} &= \beta_1 \left[\sum_{l=0}^{k+1} \tfrac{\alpha_l}{(b_2-b_1)} \Big([b_2]^{k+2-l} - [b_1]^{k+2-l}\Big)\right] + \beta_2 \left[\sum_{l=0}^{k+1} \tfrac{\alpha_l}{(b_2-b_1)} \big([b_2]^{k+1-l} - [b_1]^{k+1-l}\big)\right] + \alpha_{k+2} \\ &= \beta_1 y_{k+1} + \beta_2 \left[\sum_{l=0}^{k} \tfrac{\alpha_l}{(b_2-b_1)} \big([b_2]^{k+1-l} - [b_1]^{k+1-l}\big)\right] + \alpha_{k+2} = \beta_1 y_{k+1} + \beta_2 y_k + \alpha_{k+2}. \end{aligned} \tag{4.6}$$

Combining (4.3), (4.4), and (4.6) hence ensures that for all $k \in \mathbb{N}_0$ it holds that $y_k = x_k$. The proof of Lemma 4.1 is thus complete. $\square$

4.2 Elementary estimates for full-history recursions

Lemma 4.2. *Let $\gamma \in \{0,1\}$, $\beta \in (0,\infty)$, let $\alpha_k \in \mathbb{C}$, $k \in \mathbb{N}_0$, and $x_k \in \mathbb{C}$, $k \in \mathbb{N}_0$, satisfy for all $k \in \mathbb{N}_0$ that*

$$x_k = \alpha_k + \sum_{l=0}^{k-1} (k-l)^\gamma \beta^{(k-l)} \big[x_l + \mathbb{1}_{\mathbb{N}}(l)\, x_{\max\{l-1,0\}}\big]. \tag{4.7}$$

Then it holds for all $k \in \mathbb{N}_0$ that

$$\begin{aligned} x_k &= \sum_{l=0}^{k} \frac{[\alpha_l - \mathbb{1}_{\mathbb{N}}(l)\, 2^\gamma \beta \alpha_{\max\{l-1,0\}} + \mathbb{1}_{[2,\infty)}(l)\, \gamma \beta^2 \alpha_{\max\{l-2,0\}}]}{2^{1+\gamma(k-l)} \sqrt{5^\gamma \beta^2 + 4^\gamma \beta} \Big(\big[3^\gamma \beta + \sqrt{5^\gamma \beta^2 + 4^\gamma \beta}\big]^{k+1-l} - \big[3^\gamma \beta - \sqrt{5^\gamma \beta^2 + 4^\gamma \beta}\big]^{k+1-l} \Big)^{-1}} \\ &= \begin{cases} \displaystyle\sum_{l=0}^{k} \frac{\big[\alpha_l - \mathbb{1}_{\mathbb{N}}(l)\, \beta \alpha_{\max\{l-1,0\}}\big] \Big(\big[\beta + \sqrt{\beta^2+\beta}\big]^{k+1-l} - \big[\beta - \sqrt{\beta^2+\beta}\big]^{k+1-l} \Big)}{2\sqrt{\beta^2+\beta}} & : \gamma = 0 \\ \displaystyle\sum_{l=0}^{k} \frac{\big[\alpha_l - \mathbb{1}_{\mathbb{N}}(l)\, 2\beta \alpha_{\max\{l-1,0\}} + \mathbb{1}_{[2,\infty)}(l)\, \beta^2 \alpha_{\max\{l-2,0\}}\big]}{2^{(k+1-l)} \sqrt{5\beta^2+4\beta} \Big(\big[3\beta + \sqrt{5\beta^2+4\beta}\big]^{k+1-l} - \big[3\beta - \sqrt{5\beta^2+4\beta}\big]^{k+1-l} \Big)^{-1}} & : \gamma = 1 \end{cases}. \end{aligned} \tag{4.8}$$

Proof of Lemma 4.2. Throughout this proof let $x_{-1}, x_{-2}, x_{-3}, \alpha_{-1}, \alpha_{-2}, \alpha_{-3} \in \mathbb{C}$ satisfy that $x_{-1} = x_{-2} = x_{-3} = \alpha_{-1} = \alpha_{-2} = \alpha_{-3} = 0$. Note that (4.7) ensures that for all $k \in \mathbb{N}_0$ it holds that

$$x_k = \alpha_k + \sum_{l=0}^{k-1} (k-l)^\gamma \beta^{(k-l)} [x_l + x_{l-1}]. \tag{4.9}$$

Hence, we obtain that for all $k \in \mathbb{N}_0$ it holds that

$$\begin{aligned} x_{k+1} - \beta x_k &= \left[\alpha_{k+1} + \sum_{l=0}^{k} (k+1-l)^\gamma \beta^{(k+1-l)} [x_l + x_{l-1}]\right] - \beta \left[\alpha_k + \sum_{l=0}^{k-1} (k-l)^\gamma \beta^{(k-l)} [x_l + x_{l-1}]\right] \\ &= \alpha_{k+1} - \beta \alpha_k + \beta [x_k + x_{k-1}] + \sum_{l=0}^{k-1} \beta^{(k+1-l)} [(k+1-l)^\gamma - (k-l)^\gamma][x_l + x_{l-1}]. \end{aligned} \tag{4.10}$$

This implies that for all $k \in \mathbb{N}_0$ it holds that

$$x_{k+1} - \beta x_k = \alpha_{k+1} - \beta \alpha_k + \beta [x_k + x_{k-1}] + \mathbb{1}_{\{1\}}(\gamma) \sum_{l=0}^{k-1} \beta^{(k+1-l)} [x_l + x_{l-1}]. \tag{4.11}$$

This proves that for all $k \in \mathbb{N}_0$ it holds that

$$x_{k+1} = \alpha_{k+1} - \beta\alpha_k + 2\beta x_k + \beta\, x_{k-1} + \mathbb{1}_{\{1\}}(\gamma) \sum_{l=0}^{k-1} \beta^{(k+1-l)}[x_l + x_{l-1}]. \tag{4.12}$$

This and the fact that $x_0 = \alpha_0$ show that for all $k \in \{-1, 0, 1, 2, \dots\}$ it holds that

$$x_k = [\alpha_k - \beta\alpha_{k-1}] + 2\beta x_{k-1} + \beta x_{k-2} + \mathbb{1}_{\{1\}}(\gamma) \sum_{l=0}^{k-2} \beta^{(k-l)}[x_l + x_{l-1}]. \tag{4.13}$$

Therefore, we obtain that for all $k \in \mathbb{N}_0$ it holds that

$$\begin{aligned}
x_k - \beta x_{k-1} &= \left[[\alpha_k - \beta\alpha_{k-1}] + 2\beta x_{k-1} + \beta x_{k-2} + \mathbb{1}_{\{1\}}(\gamma) \sum_{l=0}^{k-2} \beta^{(k-l)}[x_l + x_{l-1}]\right] \\
&\quad - \beta\left[[\alpha_{k-1} - \beta\alpha_{k-2}] + 2\beta x_{k-2} + \beta x_{k-3} + \mathbb{1}_{\{1\}}(\gamma) \sum_{l=0}^{k-3} \beta^{(k-1-l)}[x_l + x_{l-1}]\right] \\
&= \left[\alpha_k - 2\beta\alpha_{k-1} + \beta^2\alpha_{k-2}\right] + 2\beta x_{k-1} + \beta[1-\beta]x_{k-2} - \beta^2 x_{k-2} - \beta^2 x_{k-3} \\
&\quad + \mathbb{1}_{\{1\}}(\gamma)\left[\sum_{l=0}^{k-2} \beta^{(k-l)}[x_l + x_{l-1}] - \sum_{l=0}^{k-3} \beta^{(k-l)}[x_l + x_{l-1}]\right] \\
&= \left[\alpha_k - 2\beta\alpha_{k-1} + \beta^2\alpha_{k-2}\right] + 2\beta x_{k-1} + \beta[1-\beta]x_{k-2} - \beta^2[x_{k-2} + x_{k-3}] \\
&\quad + \mathbb{1}_{\{1\}}(\gamma)\, \beta^2[x_{k-2} + x_{k-3}] \\
&= \left[\alpha_k - 2\beta\alpha_{k-1} + \beta^2\alpha_{k-2}\right] + 2\beta x_{k-1} + \beta[1-\beta]x_{k-2} - \mathbb{1}_{\{0\}}(\gamma)\, \beta^2[x_{k-2} + x_{k-3}].
\end{aligned} \tag{4.14}$$

This ensures that for all $k \in \mathbb{N}_0$ it holds that

$$x_k = \left[\alpha_k - 2\beta\alpha_{k-1} + \beta^2\alpha_{k-2}\right] + 3\beta x_{k-1} + \beta[1-\beta]x_{k-2} - \mathbb{1}_{\{0\}}(\gamma)\, \beta^2[x_{k-2} + x_{k-3}]. \tag{4.15}$$

Combining this, (4.13), and Lemma 4.1 yields that for all $k \in \mathbb{N}_0$ it holds that

$$x_k = \begin{cases} \sum_{l=0}^{k} \frac{\left[\alpha_l - \beta\alpha_{l-1}\right]\left(\left[2\beta + \sqrt{4\beta^2 + 4\delta\beta}\right]^{k+1-l} - \left[2\beta - \sqrt{4\beta^2 + 4\delta\beta}\right]^{k+1-l}\right)}{2^{(k+1-l)}\sqrt{4\beta^2 + 4\beta}} & : \gamma = 0 \\ \sum_{l=0}^{k} \frac{\left(\left[3\beta + \sqrt{(3\beta)^2 + 4\beta - 4\beta^2}\right]^{k+1-l} - \left[3\beta - \sqrt{(3\beta)^2 + 4\beta - 4\beta^2}\right]^{k+1-l}\right)}{\left[\alpha_l - 2\beta\alpha_{l-1} + \beta^2\alpha_{l-2}\right]^{-1} 2^{(k+1-l)}\sqrt{(3\beta)^2 + 4\beta - 4\beta^2}} & : \gamma = 1 \end{cases}. \tag{4.16}$$

This proves that for all $k \in \mathbb{N}_0$ it holds that

$$x_k = \begin{cases} \sum_{l=0}^{k} \frac{\left[\alpha_l - \beta\alpha_{l-1}\right]\left(\left[\beta + \sqrt{\beta^2 + \beta}\right]^{k+1-l} - \left[\beta - \sqrt{\beta^2 + \beta}\right]^{k+1-l}\right)}{2\sqrt{\beta^2 + \beta}} & : \gamma = 0 \\ \sum_{l=0}^{k} \frac{\left[\alpha_l - 2\beta\alpha_{l-1} + \beta^2\alpha_{l-2}\right]\left(\left[3\beta + \sqrt{5\beta^2 + 4\beta}\right]^{k+1-l} - \left[3\beta - \sqrt{5\beta^2 + 4\beta}\right]^{k+1-l}\right)}{2^{(k+1-l)}\sqrt{5\beta^2 + 4\beta}} & : \gamma = 1 \end{cases}. \tag{4.17}$$

The proof of Lemma 4.2 is thus complete. □

Corollary 4.3. *Let $\gamma \in \{0, 1\}$, $\beta \in [1, \infty)$, $\alpha_0, \alpha_1, x_0, x_1, x_2, \dots \in [0, \infty)$ satisfy for all $k \in \mathbb{N}_0$ that*

$$x_k \le \mathbb{1}_{\mathbb{N}}(k)(\alpha_0 + \alpha_1 k)\beta^k + \sum_{l=0}^{k-1} (k-l)^\gamma \beta^{(k-l)}\left[x_l + x_{\max\{l-1,0\}}\right]. \tag{4.18}$$

Then it holds for all $k \in \mathbb{N}_0$ that

$$x_k \le \frac{(\alpha_0 + \alpha_1)\beta^k \mathbb{1}_{\mathbb{N}}(k)}{(4+\gamma)^{1/2}(1 + 2^{(1+\gamma)/2})^{-k}} = \begin{cases} \mathbb{1}_{\mathbb{N}}(k)(\alpha_0 + \alpha_1)2^{-1}(1 + 2^{1/2})^k \beta^k & : \gamma = 0 \\ \mathbb{1}_{\mathbb{N}}(k)(\alpha_0 + \alpha_1)5^{-1/2}(3\beta)^k & : \gamma = 1 \end{cases}. \tag{4.19}$$

Proof of Corollary 4.3. Throughout this proof let $A\colon \mathbb{N}_0 \to [0,\infty)$ satisfy for all $k \in \mathbb{N}_0$ that $A_k = \mathbb{1}_{\mathbb{N}}(k)(\alpha_0+\alpha_1 k)\beta^k$. Note that (4.18) ensures that $x_0 = 0$. This assures that for all $k \in \mathbb{N}_0$ it holds that

$$\begin{aligned} x_k &\leq \mathbb{1}_{\mathbb{N}}(k)(\alpha_0+\alpha_1 k)\beta^k + \sum_{l=0}^{k-1}(k-l)^\gamma \beta^{(k-l)}\big[x_l + x_{\max\{l-1,0\}}\big] \\ &= \mathbb{1}_{\mathbb{N}}(k)(\alpha_0+\alpha_1 k)\beta^k + \sum_{l=0}^{k-1}(k-l)^\gamma \beta^{(k-l)}\big[x_l + \mathbb{1}_{\mathbb{N}}(l)\, x_{\max\{l-1,0\}}\big] \end{aligned} \tag{4.20}$$

Next observe that for all $l \in \mathbb{N}_0$ it holds that

$$\begin{aligned} A_l - \mathbb{1}_{\mathbb{N}}(l)\beta A_{\max\{l-1,0\}} &= \big[\mathbb{1}_{\mathbb{N}}(l)(\alpha_0+\alpha_1 l) - \mathbb{1}_{\mathbb{N}}(l)\mathbb{1}_{\mathbb{N}}(l-1)(\alpha_0+\alpha_1(l-1))\big]\beta^l \\ &= \mathbb{1}_{\{1\}}(l)(\alpha_0+\alpha_1)\beta \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} &A_l - \mathbb{1}_{\mathbb{N}}(l)2\beta A_{\max\{l-1,0\}} + \mathbb{1}_{[2,\infty)}(l)\beta^2 A_{\max\{l-2,0\}} \\ &= \big[\mathbb{1}_{\mathbb{N}}(l)(\alpha_0+\alpha_1 l) - \mathbb{1}_{\mathbb{N}}(l)2\mathbb{1}_{\mathbb{N}}(l-1)(\alpha_0+\alpha_1(l-1)) + \mathbb{1}_{\mathbb{N}}(l-2)(\alpha_0+\alpha_1(l-2))\big]\beta^l \\ &= \mathbb{1}_{\{1\}}(l)(\alpha_0+\alpha_1)\beta - \mathbb{1}_{\{2\}}(l)\alpha_0\beta^2. \end{aligned} \tag{4.22}$$

Lemma 4.2 therefore proves that for all $k \in \mathbb{N}_0$ it holds that

$$\begin{aligned} x_k &\leq \mathbb{1}_{\mathbb{N}}(k)(\alpha_0+\alpha_1)\beta\frac{\big[\beta+\sqrt{\beta^2+\beta}\big]^k - \big[\beta-\sqrt{\beta^2+\beta}\big]^k}{2\sqrt{\beta^2+\beta}} \\ &\leq \mathbb{1}_{\mathbb{N}}(k)(\alpha_0+\alpha_1)\beta\frac{\big[\beta+\sqrt{\beta^2+\beta}\big]^k}{2\sqrt{\beta^2+\beta}} \leq \mathbb{1}_{\mathbb{N}}(k)\frac{(\alpha_0+\alpha_1)}{2}(1+\sqrt{2})^k\beta^k \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} x_k &\leq \mathbb{1}_{\mathbb{N}}(k)(\alpha_0+\alpha_1)\beta\frac{\Big[3\beta+\sqrt{5\beta^2+4\beta}\Big]^k - \Big[3\beta-\sqrt{5\beta^2+4\beta}\Big]^k}{2^k\sqrt{5\beta^2+4\beta}} \\ &\leq \mathbb{1}_{\mathbb{N}}(k)(\alpha_0+\alpha_1)\beta\frac{\Big[3\beta+\sqrt{5\beta^2+4\beta}\Big]^k}{2^k\sqrt{5\beta^2+4\beta}} = \mathbb{1}_{\mathbb{N}}(k)(\alpha_0+\alpha_1)\frac{\beta^k\Big[3+\sqrt{5+\frac{4}{\beta}}\Big]^k}{2^k\sqrt{5+\frac{4}{\beta}}} \\ &\leq \mathbb{1}_{\mathbb{N}}(k)\frac{(\alpha_0+\alpha_1)}{\sqrt{5}}(3\beta)^k. \end{aligned} \tag{4.24}$$

Combining (4.23) and (4.24) hence establishes (4.19). The proof of Corollary 4.3 is thus complete. □

4.3 Complexity analysis in the case of stochastic fixed-point equations

Proposition 4.4. *Let $T, L, p, q, \alpha, \beta, \mathfrak{d}, \mathfrak{B} \in [0,\infty)$, $m_1, m_2, m_3, \ldots \in \mathbb{N}$, $\Theta = \bigcup_{n\in\mathbb{N}}\mathbb{Z}^n$, let $f_d \in C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, $d \in \mathbb{N}$, and let $g_d \in C(\mathbb{R}^d,\mathbb{R})$, $d \in \mathbb{N}$, assume for all $d \in \mathbb{N}$, $t \in [0,T]$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that*

$$\liminf_{j\to\infty} m_j = \infty, \qquad m_{d+1} \leq \mathfrak{B} m_d, \qquad |f_d(t,x,v) - f_d(t,x,w)| \leq L|v-w|, \tag{4.25}$$

`Sequence of m_j's not necessary`

and $\max\{|f_d(t,x,0)|, |g_d(x)|\} \leq L d^p(1+\sum_{k=1}^d |x_k|)^q$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\mathfrak{u}^\theta\colon \Omega \to [0,1]$, $\theta \in \Theta$, be i.i.d. random variables, assume for all $r \in (0,1)$ it holds that $\mathbb{P}(\mathfrak{u}^0 \leq r) = r$, let $\mathcal{U}^\theta\colon [0,T]\times\Omega \to [0,T]$, $\theta \in \Theta$, satisfy for all $t \in [0,T]$, $\theta \in \Theta$ that $\mathcal{U}^\theta_t = t + (T-t)\mathfrak{u}^\theta$, let

$W^{d,\theta}\colon [0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $\theta\in\Theta$, be independent standard Brownian motions, assume for every $d\in\mathbb{N}$ that $(\mathcal{U}^\theta)_{\theta\in\Theta}$ and $(W^{d,\theta})_{\theta\in\Theta}$ are independent, let $u_d\in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, $d\in\mathbb{N}$, satisfy for all $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $\mathbb{E}[|g_d(x+W^{d,0}_{T-t})|+\int_t^T|f_d(s,x+W^{d,0}_{s-t},u_d(s,x+W^{d,0}_{s-t}))|\,ds]<\infty$ and

$$u_d(t,x)=\mathbb{E}\big[g_d(x+W^{d,0}_{T-t})\big]+\int_t^T\mathbb{E}\big[f_d(s,x+W^{d,0}_{s-t},u_d(s,x+W^{d,0}_{s-t}))\big]\,ds, \tag{4.26}$$

let $U^{d,\theta}_{n,j}\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $d,j,n\in\mathbb{Z}$, $\theta\in\Theta$, satisfy for all $n\in\mathbb{N}_0$, $d,j\in\mathbb{N}$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that

$$\begin{aligned}U^{d,\theta}_{n,j}(t,x)&=\frac{\mathbb{1}_{\mathbb{N}}(n)}{(m_j)^n}\left[\sum_{k=1}^{(m_j)^n}g_d\big(x+W^{d,(\theta,0,-k)}_{T-t}\big)\right]\\&\quad+\sum_{i=0}^{n-1}\frac{(T-t)}{(m_j)^{n-i}}\left[\sum_{k=1}^{(m_j)^{n-i}}\Big[f_d\big(\mathcal{U}^{(\theta,i,k)}_t,x+W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t-t},U^{d,(\theta,i,k)}_{i,j}\big(\mathcal{U}^{(\theta,i,k)}_t,x+W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t-t}\big)\big)\right.\\&\quad\left.-\mathbb{1}_{\mathbb{N}}(i)\,f_d\big(\mathcal{U}^{(\theta,i,k)}_t,x+W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t-t},U^{d,(\theta,-i,k)}_{i-1,j}\big(\mathcal{U}^{(\theta,i,k)}_t,x+W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t-t}\big)\big)\Big]\right],\end{aligned} \tag{4.27}$$

and let $\mathfrak{C}_{d,n,j}\in\mathbb{R}$, $d,n,j\in\mathbb{N}_0$, satisfy for all $d,j\in\mathbb{N}$, $n\in\mathbb{N}_0$ that

$$\mathfrak{C}_{d,n,j}\le\mathbb{1}_{\mathbb{N}}(n)\,\alpha d^{\mathfrak{d}}(m_j)^n+\sum_{k=0}^{n-1}(m_j)^{n-k}\big(\alpha d^{\mathfrak{d}}+\mathfrak{C}_{d,k,j}+\mathfrak{C}_{d,\max\{k-1,0\},j}\big). \tag{4.28}$$

Then there exist $\mathcal{n}\colon\mathbb{N}\times\mathbb{R}\to\mathbb{N}$ and $c=(c_{\mathfrak{p},\delta})_{(\mathfrak{p},\delta)\in\mathbb{R}^2}\colon\mathbb{R}^2\to\mathbb{R}$ such that for all $d\in\mathbb{N}$, $\varepsilon,\delta\in(0,\infty)$, $\mathfrak{p}\in[2,\infty)$ with $\limsup_{j\to\infty}[{}^{(m_j)^{\mathfrak{p}/2}}/{}_j]<\infty$ it holds that

$$\begin{aligned}(\mathcal{n}(d,\varepsilon))^\beta\mathfrak{C}_{d,\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)}&\le\alpha d^{\mathfrak{d}}(\mathcal{n}(d,\varepsilon))^\beta(1+\sqrt{2})^{\mathcal{n}(d,\varepsilon)}\big(m_{\mathcal{n}(d,\varepsilon)}\big)^{\mathcal{n}(d,\varepsilon)}\\&\le\alpha c_{\mathfrak{p},\delta}d^{\mathfrak{d}+(p+q)(2+\delta)}(\min\{1,\varepsilon\})^{-(2+\delta)}\end{aligned} \tag{4.29}$$

and

`Supremum over time and space`

$$\sup_{t\in[0,T]}\sup_{x\in[-L,L]^d}\Big(\mathbb{E}\Big[\big|u_d(t,x)-U^{d,0}_{\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)}(t,x)\big|^{\mathfrak{p}}\Big]\Big)^{1/\mathfrak{p}}\le\varepsilon. \tag{4.30}$$

Proof of Proposition 4.4. Throughout this proof let $\mathfrak{m}_{\mathfrak{p}}=\mathfrak{K}_{\mathfrak{p}}\sqrt{\mathfrak{p}-1}$, $\mathfrak{p}\in[2,\infty)$, let $\mathbb{F}^d_t\subseteq\mathcal{F}$, $d\in\mathbb{N}$, $t\in[0,T]$, satisfy for all $d\in\mathbb{N}$, $t\in[0,T]$ that

`May or may not be necessary, as it is a sigma algebra to calculate F`

$$\mathbb{F}^d_t=\begin{cases}\bigcap_{s\in(t,T]}\sigma\big(\sigma(W^{d,0}_r\colon r\in[0,s])\cup\{A\in\mathcal{F}\colon\mathbb{P}(A)=0\}\big)&\colon t<T\\\sigma\big(\sigma(W^{d,0}_s\colon s\in[0,T])\cup\{A\in\mathcal{F}\colon\mathbb{P}(A)=0\}\big)&\colon t=T\end{cases}, \tag{4.31}$$

let $a_d\in C([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$, $d\in\mathbb{N}$, satisfy for all $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $a_d(t,x)=0$, let $b_d\in C([0,T]\times\mathbb{R}^d,\mathbb{R}^{d\times d})$, $d\in\mathbb{N}$, satisfy for all $d\in\mathbb{N}$, $t\in[0,T]$, $x,v\in\mathbb{R}^d$ that $b_d(t,x)v=v$, let $\eta_{d,\mathfrak{p}}\in[0,\infty)$, $d\in\mathbb{N}$, $\mathfrak{p}\in[2,\infty)$, satisfy for all $\mathfrak{p}\in[2,\infty)$, $d\in\mathbb{N}$ that

$$\eta_{d,\mathfrak{p}}=\mathfrak{m}_{\mathfrak{p}}L2^{\max\{q,1\}}d^{p+q}\big((1+L^2)^{q/2}+(q\mathfrak{p}+1)^{1/\mathfrak{p}}\big)\exp\big(\tfrac{[q(q\mathfrak{p}+3)+1]T}{2}+(L+1)T\big), \tag{4.32}$$

and let $\mathcal{n}\colon\mathbb{N}\times\mathbb{R}\to[1,\infty]$ satisfy for all $d\in\mathbb{N}$, $\varepsilon\in(0,\infty)$ that

$$\begin{aligned}&\mathcal{n}(d,\varepsilon)=\\&\inf\left(\left\{n\in\mathbb{N}\colon\left[\sup\left\{\eta_{d,\mathfrak{p}}\left[\frac{(1+2LT)\exp\big(\frac{(m_n)^{\mathfrak{p}/2}}{n}\big)}{(m_n)^{1/2}}\right]^n\le\varepsilon\colon\begin{array}{c}\mathfrak{p}\in[2,\infty)\text{ with}\\\limsup_{j\to\infty}{}^{(m_j)^{\mathfrak{p}/2}}/{}_j<\infty\end{array}\right\}\right]\right\}\cup\{\infty\}\right)\end{aligned} \tag{4.33}$$

(cf. Definition 3.1). Observe that (4.31) guarantees that $\mathbb{F}^d_t\subseteq\mathcal{F}$, $d\in\mathbb{N}$, $t\in[0,T]$, satisfies that

(I) it holds for all $d \in \mathbb{N}$ that $\{A \in \mathcal{F} \colon \mathbb{P}(A) = 0\} \subseteq \mathbb{F}_0^d$ and

(II) it holds for all $d \in \mathbb{N}$, $t \in [0,T]$ that $\mathbb{F}_t^d = \cap_{s \in (t,T]} \mathbb{F}_s^d$.

Combining item (I), item (II), (4.31), and Hutzenthaler et al. [31, Lemma 2.17] therefore assures that for all $d \in \mathbb{N}$ it holds that $W^{d,0} \colon [0,T] \times \Omega \to \mathbb{R}^d$ is a standard $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t^d)_{t \in [0,T]})$-Brownian motion. In addition, note that (4.31) ensures that for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ it holds that $[0,T] \times \Omega \ni (t,\omega) \mapsto x + W_t^{d,0}(\omega) \in \mathbb{R}^d$ is an $(\mathbb{F}_t^d)_{t \in [0,T]}/\mathcal{B}(\mathbb{R}^d)$-adapted stochastic process with continuous sample paths. This, the fact that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $a_d(t,x) = 0$, and the fact that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x, v \in \mathbb{R}^d$ it holds that $b_d(t,x)v = v$ yield that for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ it holds that $[0,T] \times \Omega \ni (t,\omega) \mapsto x + W_t^{d,0}(\omega) \in \mathbb{R}^d$ satisfies for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$x + W_t^{d,0} = x + \int_0^t 0 \, ds + \int_0^t dW_s^{d,0} = x + \int_0^t a_d(s, x + W_s^{d,0}) \, ds + \int_0^t b_d(s, x + W_s^{d,0}) \, dW_s^{d,0}. \tag{4.34}$$

Combining this and Hutzenthaler et al. [31, Lemma 2.6] (applied for every $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ with $d \curvearrowleft d$, $m \curvearrowleft d$, $T \curvearrowleft T$, $C_1 \curvearrowleft d$, $C_2 \curvearrowleft 0$, $\mathbb{F} \curvearrowleft \mathbb{F}^d$, $\xi \curvearrowleft x$, $\mu \curvearrowleft a_d$, $\sigma \curvearrowleft b_d$, $W \curvearrowleft W^{d,0}$, $X \curvearrowleft ([0,T] \times \Omega \ni (t,\omega) \mapsto x + W_t^{d,0}(\omega) \in \mathbb{R}^d)$ in the notation of [31, Lemma 2.6]) ensures that for all $r \in [0,\infty)$, $d \in \mathbb{N}$, $x \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\|x + W_t^{d,0}\|^r\big] \leq \max\{T,1\}\big((1 + \|x\|^2)^{r/2} + (r+1)d^{r/2}\big)\exp\big(\tfrac{r(r+3)T}{2}\big) < \infty \tag{4.35}$$

(cf. Definition 2.4). This, the triangle inequality, and the fact that for all $v, w \in [0,\infty)$, $r \in (0,1]$ it holds that $(v+w)^r \leq v^r + w^r$ assure that for all $\mathfrak{p} \in [2,\infty)$, $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ it holds that

Don't we already know what 1+ || x+W|| is?

$$\begin{aligned}
&\sup_{s \in [0,T]} \big(\mathbb{E}\big[(1 + \|x + W_s^{d,0}\|^q)^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} \leq 1 + \sup_{s \in [0,T]} \big(\mathbb{E}\big[\|x + W_s^{d,0}\|^{q\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} \\
&\leq 1 + \sup_{s \in [0,T]} \Big(\max\{T,1\}\big((1 + \|x\|^2)^{q\mathfrak{p}/2} + (p\mathfrak{p}+1)d^{q\mathfrak{p}/2}\big)\exp\big(\tfrac{q\mathfrak{p}(q\mathfrak{p}+3)T}{2}\big)\Big)^{1/\mathfrak{p}} \\
&\leq 1 + \max\{T^{1/\mathfrak{p}},1\}\big((1 + \|x\|^2)^{q/2} + (q\mathfrak{p}+1)^{1/\mathfrak{p}}d^{q/2}\big)\exp\big(\tfrac{q(q\mathfrak{p}+3)T}{2}\big) \\
&\leq 2\big((1 + \|x\|^2)^{q/2} + (q\mathfrak{p}+1)^{1/\mathfrak{p}}d^{q/2}\big)\exp\big(\tfrac{q(q\mathfrak{p}+3)T}{2} + \tfrac{T}{\mathfrak{p}}\big) \\
&\leq 2\big((1 + \|x\|^2)^{q/2} + (q\mathfrak{p}+1)^{1/\mathfrak{p}}d^{q/2}\big)\exp\big(\tfrac{[q(q\mathfrak{p}+3)+1]T}{2}\big) < \infty.
\end{aligned} \tag{4.36}$$

Combining this, (4.32), and the fact that for all $d \in \mathbb{N}$, $x \in [-L,L]^d$ it holds that $\|x\| \leq Ld^{1/2}$ demonstrates that for all $\mathfrak{p} \in [2,\infty)$, $d \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\mathfrak{m}_{\mathfrak{p}} L 2^{\max\{q-1,0\}} d^{p+q/2}(T+1)\exp(LT)\left[\sup_{x \in [-L,L]^d} \sup_{s \in [0,T]} \big(\mathbb{E}\big[(1 + \|x + W_s^{d,0}\|^q)^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}}\right] \\
&\leq \mathfrak{m}_{\mathfrak{p}} L 2^{\max\{q-1,0\}} d^{p+q/2}\exp(LT+T) \\
&\quad \cdot \left[\sup_{x \in [-L,L]^d} \Big[2\big((1 + \|x\|^2)^{q/2} + (q\mathfrak{p}+1)^{1/\mathfrak{p}}d^{q/2}\big)\exp\big(\tfrac{[q(q\mathfrak{p}+3)+1]T}{2}\big)\Big]\right] \\
&\leq \mathfrak{m}_{\mathfrak{p}} L 2^{\max\{q,1\}} d^{p+q/2}\big((1 + L^2 d)^{q/2} + (q\mathfrak{p}+1)^{1/\mathfrak{p}}d^{q/2}\big)\exp\big(\tfrac{[q(q\mathfrak{p}+3)+1]T}{2} + (L+1)T\big) \\
&\leq \mathfrak{m}_{\mathfrak{p}} L 2^{\max\{q,1\}} d^{p+q}\big((1 + L^2)^{q/2} + (q\mathfrak{p}+1)^{1/\mathfrak{p}}\big)\exp\big(\tfrac{[q(q\mathfrak{p}+3)+1]T}{2} + (L+1)T\big) = \eta_{d,\mathfrak{p}} < \infty.
\end{aligned} \tag{4.37}$$

This and (4.25) guarantee that for all $\mathfrak{p} \in [2,\infty)$ which satisfy $\limsup_{j \to \infty} [(m_j)^{\mathfrak{p}/2}/j] < \infty$ it holds that

$$\limsup_{n \to \infty} \eta_{d,\mathfrak{p}} \left[\frac{(1 + 2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)}{(m_n)^{1/2}}\right]^n = 0. \tag{4.38}$$

Combining this and (4.33) implies that for all $d \in \mathbb{N}$, $\varepsilon \in (0,\infty)$ it holds that $\mathfrak{n}(d,\varepsilon) \in \mathbb{N}$. Next observe that the fact that for all $m \in \mathbb{N}$, $r, v_1, v_2, \ldots, v_m \in [0,\infty)$ it holds that $[\sum_{k=1}^m v_k]^r \leq$

$m^{\max\{r-1,0\}}[\sum_{k=1}^m v_k^r]$ and the hypothesis that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ it holds that $\max\{|f_d(t,x,0)|, |g_d(x)|\} \leq Ld^p(1 + \sum_{k=1}^d |x_k|)^q$ ensure that for all $d \in \mathbb{N}$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\max\{|f_d(t,x,0)|, |g_d(x)|\} \leq Ld^p\big(1 + \textstyle\sum_{k=1}^d |x_k|\big)^q \leq L2^{\max\{q-1,0\}} d^p \big[1 + \big(\textstyle\sum_{k=1}^d |x_k|\big)^q\big] \\
&\leq L2^{\max\{q-1,0\}} d^p \big[1 + \big(d^{2-1} \textstyle\sum_{k=1}^d |x_k|^2\big)^{q/2}\big] \leq L2^{\max\{q-1,0\}} d^{p+q/2} \big(1 + \|x\|^q\big).
\end{aligned} \tag{4.39}$$

This, (4.25), and Corollary 3.15 (applied for every $\mathfrak{p} \in [2,\infty)$, $d, j \in \mathbb{N}$ with $m \curvearrowleft m_j$, $\mathfrak{m} \curvearrowleft \mathfrak{m}_\mathfrak{p}$, $p \curvearrowleft q$, $f \curvearrowleft f_d$, $g \curvearrowleft g_d$, $T \curvearrowleft T$, $L \curvearrowleft L$, $\mathfrak{L} \curvearrowleft L2^{\max\{q-1,0\}} d^{p+q/2}$, $\mathfrak{u}^\theta \curvearrowleft \mathfrak{u}^\theta$, $\mathcal{U}^\theta \curvearrowleft \mathcal{U}^\theta$, $W^\theta \curvearrowleft W^{d,\theta}$, $u \curvearrowleft u_d$, $U_n^0 \curvearrowleft U_{n,j}^{d,0}$ in the notation of Corollary 3.15) assure that for all $\mathfrak{p} \in [2,\infty)$, $n \in \mathbb{N}_0$, $d, j \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
&\big(\mathbb{E}\big[|u_d(t,x) - U_{n,j}^{d,0}(t,x)|^\mathfrak{p}\big]\big)^{1/\mathfrak{p}} \\
&\leq \frac{\mathfrak{m}_\mathfrak{p} L2^{\max\{q-1,0\}} d^{p+q/2} (T+1) \exp(LT) \big(1 + 2LT\big)^n}{(m_j)^{n/2} \exp\big(-{(m_j)^{\mathfrak{p}/2}}/{\mathfrak{p}}\big)} \left[\sup_{s\in[0,T]} \big(\mathbb{E}\big[(1 + \|x + W_s^{d,0}\|^q)^\mathfrak{p}\big]\big)^{1/\mathfrak{p}}\right].
\end{aligned} \tag{4.40}$$

Combining this, (4.32), (4.36), and (4.37) demonstrates that for all $\mathfrak{p} \in [2,\infty)$, $n \in \mathbb{N}_0$, $d, j \in \mathbb{N}$, $t \in [0,T]$, $x \in [-L,L]^d$ it holds that

$$\big(\mathbb{E}\big[|u_d(t,x) - U_{n,j}^{d,0}(t,x)|^\mathfrak{p}\big]\big)^{1/\mathfrak{p}} \leq \eta_{d,\mathfrak{p}} \left[\frac{\big(1 + 2LT\big)^n \exp\big({(m_j)^{\mathfrak{p}/2}}/{\mathfrak{p}}\big)}{(m_j)^{n/2}}\right]. \tag{4.41}$$

This, (4.33), and the fact that for all $d \in \mathbb{N}$, $\varepsilon \in (0,\infty)$ it holds that $\mathcal{n}(d,\varepsilon) \in \mathbb{N}$ prove that for all $\mathfrak{p} \in [2,\infty)$ with $\limsup_{j\to\infty}[{(m_j)^{\mathfrak{p}/2}}/{j}] < \infty$, $d \in \mathbb{N}$, $t \in [0,T]$, $x \in [-L,L]^d$, $\varepsilon \in (0,\infty)$ it holds that

$$\begin{aligned}
\big(\mathbb{E}\big[|u_d(t,x) - U_{\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)}^{d,0}(t,x)|^\mathfrak{p}\big]\big)^{1/\mathfrak{p}} &\leq \eta_{d,\mathfrak{p}} \left[\frac{\big(1 + 2LT\big)^{\mathcal{n}(d,\varepsilon)} \exp\big((m_{\mathcal{n}(d,\varepsilon)})^{\mathfrak{p}/2}\big)}{(m_{\mathcal{n}(d,\varepsilon)})^{\mathcal{n}(d,\varepsilon)/2}}\right] \\
&\leq \eta_{d,\mathfrak{p}} \left[\frac{\big(1 + 2LT\big) \exp\big({(m_{\mathcal{n}(d,\varepsilon)})^{\mathfrak{p}/2}}/{\mathcal{n}(d,\varepsilon)}\big)}{(m_{\mathcal{n}(d,\varepsilon)})^{1/2}}\right]^{\mathcal{n}(d,\varepsilon)} \leq \varepsilon.
\end{aligned} \tag{4.42}$$

This establishes (4.29). Next note that (4.28) implies that for all $d, j \in \mathbb{N}$, $n \in \mathbb{N}_0$ it holds that

$$\begin{aligned}
\mathfrak{C}_{d,n,j} &\leq \mathbb{1}_\mathbb{N}(n)\, \alpha d^\mathfrak{d} (m_j)^n + \sum_{k=0}^{n-1} (m_j)^{n-k} \alpha d^\mathfrak{d} + \sum_{k=0}^{n-1} (m_j)^{n-k} \big(\mathfrak{C}_{d,k,j} + \mathfrak{C}_{d,\max\{k-1,0\},j}\big) \\
&\leq \mathbb{1}_\mathbb{N}(n) \big(\alpha d^\mathfrak{d} + n\alpha d^\mathfrak{d}\big) (m_j)^n + \sum_{k=0}^{n-1} (m_j)^{n-k} \big(\mathfrak{C}_{d,k,j} + \mathfrak{C}_{d,\max\{k-1,0\},j}\big).
\end{aligned} \tag{4.43}$$

Combining this and Corollary 4.3 (applied for every $d, j \in \mathbb{N}$ with $\gamma \curvearrowleft 0$, $\beta \curvearrowleft m_j$, $\alpha_0 \curvearrowleft \alpha d^\mathfrak{d}$, $\alpha_1 \curvearrowleft \alpha d^\mathfrak{d}$, $(x_n)_{n\in\mathbb{N}_0} \curvearrowleft (\mathfrak{C}_{d,n,j})_{n\in\mathbb{N}_0}$ in the notation of Corollary 4.3) guarantees that for all $n \in \mathbb{N}_0$, $d, j \in \mathbb{N}$ it holds that

$$\mathfrak{C}_{d,n,j} \leq \mathbb{1}_\mathbb{N}(n)\, \alpha d^\mathfrak{d} (1 + \sqrt{2})^n (m_j)^n. \tag{4.44}$$

Furthermore, observe that (4.33), the fact that for all $j \in \mathbb{N}$ it holds that $m_j \in \mathbb{N}$, and the fact that for all $\mathfrak{p} \in [2,\infty)$, $d \in \mathbb{N}$ it holds that $\eta_{d,\mathfrak{p}} \in [0,\infty)$ ensure that for all $\mathfrak{p} \in [2,\infty)$, $d \in \mathbb{N}$, $\varepsilon \in (0,\infty)$ with $\limsup_{j\to\infty}[{(m_j)^{\mathfrak{p}/2}}/{j}] < \infty$ and $\mathcal{n}(d,\varepsilon) \in \mathbb{N} \cap [2,\infty)$ it holds that

$$\eta_{d,\mathfrak{p}} \left[\frac{\big(1 + 2LT\big) \exp\big({(m_{\mathcal{n}(d,\varepsilon)-1})^{\mathfrak{p}/2}}/{(\mathcal{n}(d,\varepsilon)-1)}\big)}{(m_{\mathcal{n}(d,\varepsilon)-1})^{1/2}}\right]^{(\mathcal{n}(d,\varepsilon)-1)} > \min\{1,\varepsilon\}. \tag{4.45}$$

Combining this and (4.44) demonstrates that for all $\mathfrak{p} \in [2, \infty)$, $d \in \mathbb{N}$, $\varepsilon, \delta \in (0, \infty)$ with $\limsup_{j\to\infty} [(m_j)^{\mathfrak{p}/2}/j] < \infty$ and $\mathscr{n}(d, \varepsilon) \in \mathbb{N} \cap [2, \infty)$ it holds that

$$\begin{aligned}
&(\mathscr{n}(d,\varepsilon))^{\beta}\mathfrak{C}_{d,\mathscr{n}(d,\varepsilon),\mathscr{n}(d,\varepsilon)} \leq \mathbb{1}_{\mathbb{N}}(\mathscr{n}(d,\varepsilon))\,\alpha d^{\mathfrak{d}}(1+\sqrt{2})^{\mathscr{n}(d,\varepsilon)}(\mathscr{n}(d,\varepsilon))^{\beta}\big(m_{\mathscr{n}(d,\varepsilon)}\big)^{\mathscr{n}(d,\varepsilon)} \\
&\leq \frac{\alpha d^{\mathfrak{d}}\big((1+\sqrt{2})m_{\mathscr{n}(d,\varepsilon)}\big)^{\mathscr{n}(d,\varepsilon)}}{(\mathscr{n}(d,\varepsilon))^{-\beta}}\left[\frac{\eta_{d,\mathfrak{p}}}{\min\{1,\varepsilon\}}\left[\frac{(1+2LT)\exp\big((m_{\mathscr{n}(d,\varepsilon)-1})^{\mathfrak{p}/2}/(\mathscr{n}(d,\varepsilon)-1)\big)}{(m_{\mathscr{n}(d,\varepsilon)-1})^{1/2}}\right]^{(\mathscr{n}(d,\varepsilon)-1)}\right]^{(2+\delta)} \\
&= \frac{\alpha d^{\mathfrak{d}}(\eta_{d,\mathfrak{p}})^{(2+\delta)}}{(\min\{1,\varepsilon\})^{(2+\delta)}}\left[\frac{(\mathscr{n}(d,\varepsilon))^{\beta}\big((1+2LT)\exp\big((m_{\mathscr{n}(d,\varepsilon)-1})^{\mathfrak{p}/2}/(\mathscr{n}(d,\varepsilon)-1)\big)\big)^{(\mathscr{n}(d,\varepsilon)-1)(2+\delta)}}{\big((1+\sqrt{2})m_{\mathscr{n}(d,\varepsilon)}\big)^{-\mathscr{n}(d,\varepsilon)}(m_{\mathscr{n}(d,\varepsilon)-1})^{(\mathscr{n}(d,\varepsilon)-1)(1+\delta/2)}}\right] \qquad (4.46)\\
&\leq \frac{\alpha d^{\mathfrak{d}}(\eta_{d,\mathfrak{p}})^{(2+\delta)}}{(\min\{1,\varepsilon\})^{(2+\delta)}}\left[\sup_{n\in\mathbb{N}}\left(\frac{(n+1)^{\beta}\big((1+\sqrt{2})m_{n+1}\big)^{(n+1)}\big((1+2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)\big)^{n(2+\delta)}}{(m_n)^{n(1+\delta/2)}}\right)\right] \\
&\leq \frac{\alpha d^{\mathfrak{d}}(1+\sqrt{2})(\eta_{d,\mathfrak{p}})^{(2+\delta)}}{(\min\{1,\varepsilon\})^{(2+\delta)}}\left[\sup_{n\in\mathbb{N}}\left(\frac{(n+1)^{\beta}m_{n+1}(m_{n+1})^{n}\big((1+\sqrt{2})(1+2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)\big)}{(m_n)^{n}(m_n)^{n(\delta/2)}}\right)\right].
\end{aligned}$$

This and (4.25) ensure that for all $\mathfrak{p} \in [2, \infty)$, $d \in \mathbb{N}$, $\varepsilon, \delta \in (0, \infty)$ with $\limsup_{j\to\infty} [(m_j)^{\mathfrak{p}/2}/j] < \infty$ and $\mathscr{n}(d, \varepsilon) \in \mathbb{N} \cap [2, \infty)$ it holds that

$$\begin{aligned}
&(\mathscr{n}(d,\varepsilon))^{\beta}\mathfrak{C}_{d,\mathscr{n}(d,\varepsilon),\mathscr{n}(d,\varepsilon)} \qquad (4.47)\\
&\leq \frac{\alpha d^{\mathfrak{d}}(1+\sqrt{2})(\eta_{d,\mathfrak{p}})^{(2+\delta)}}{(\min\{1,\varepsilon\})^{(2+\delta)}}\left[\sup_{n\in\mathbb{N}}\left(\frac{(n+1)^{\beta}\mathfrak{B}^{2n+1}m_1\big((1+\sqrt{2})(1+2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)\big)^{n(2+\delta)}}{(m_n)^{n(\delta/2)}}\right)\right] \\
&\leq \frac{\alpha d^{\mathfrak{d}}m_1(1+\sqrt{2})\mathfrak{B}(\eta_{d,\mathfrak{p}})^{(2+\delta)}}{(\min\{1,\varepsilon\})^{(2+\delta)}}\left[\sup_{n\in\mathbb{N}}\left(\frac{(n+1)^{\beta/n}\big((1+\sqrt{2})\mathfrak{B}(1+2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)\big)^{(2+\delta)}}{(m_n)^{\delta/2}}\right)^{n}\right].
\end{aligned}$$

Moreover, note that (4.25), (4.28), and the fact that for all $\mathfrak{p} \in [2, \infty)$, $d \in \mathbb{N}$ it holds that $\eta_{d,\mathfrak{p}} \in [0, \infty)$ ensure that for all $\mathfrak{p} \in [2, \infty)$, $d \in \mathbb{N}$, $\varepsilon, \delta \in (0, \infty)$ with $\limsup_{j\to\infty} [(m_j)^{\mathfrak{p}/2}/j] < \infty$ it holds that

$$\begin{aligned}
\mathfrak{C}_{d,1,1} &\leq \alpha d^{\mathfrak{d}}m_1 + m_1(\alpha d^{\mathfrak{d}} + \mathfrak{C}_{d,0,0} + \mathfrak{C}_{d,0,0}) \leq 2\alpha d^{\mathfrak{d}}m_1 \leq \alpha d^{\mathfrak{d}}(1+\sqrt{2})\mathfrak{B}m_1 \\
&\leq \frac{\alpha d^{\mathfrak{d}}m_1(1+\sqrt{2})\mathfrak{B}(\max\{1,\eta_{d,\mathfrak{p}}\})^{(2+\delta)}}{(\min\{1,\varepsilon\})^{(2+\delta)}} \qquad (4.48)\\
&\quad\cdot\left[\sup_{n\in\mathbb{N}}\left(\frac{(n+1)^{\beta/n}\big((1+\sqrt{2})\mathfrak{B}(1+2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)\big)^{(2+\delta)}}{(m_n)^{\delta/2}}\right)^{n}\right].
\end{aligned}$$

Combining this and (4.47) demonstrates that for all $\mathfrak{p} \in [2, \infty)$, $d \in \mathbb{N}$, $\varepsilon, \delta \in (0, \infty)$ with $\limsup_{j\to\infty} [(m_j)^{\mathfrak{p}/2}/j] < \infty$ it holds that

$$\begin{aligned}
&(\mathscr{n}(d,\varepsilon))^{\beta}\mathfrak{C}_{d,\mathscr{n}(d,\varepsilon),\mathscr{n}(d,\varepsilon)} \qquad (4.49)\\
&\leq \frac{\alpha d^{\mathfrak{d}}m_1(1+\sqrt{2})\mathfrak{B}(\max\{1,\eta_{d,\mathfrak{p}}\})^{(2+\delta)}}{(\min\{1,\varepsilon\})^{(2+\delta)}}\left[\sup_{n\in\mathbb{N}}\left(\frac{(n+1)^{\beta/n}\big(\mathfrak{B}(1+2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)\big)^{(2+\delta)}}{(1+\sqrt{2})^{-1}(m_n)^{\delta/2}}\right)^{n}\right].
\end{aligned}$$

This, the fact that $m_1 \in \mathbb{N}$, (4.37), (4.38), and (4.47) prove that for all $\mathfrak{p} \in [2, \infty)$, $d \in \mathbb{N}$, $\varepsilon, \delta \in (0, \infty)$ with $\limsup_{j\to\infty} [(m_j)^{\mathfrak{p}/2}/j] < \infty$ it holds that

$$\begin{aligned}
&(\mathscr{n}(d,\varepsilon))^{\beta}\mathfrak{C}_{d,\mathscr{n}(d,\varepsilon),\mathscr{n}(d,\varepsilon)} \\
&\leq \frac{\alpha d^{\mathfrak{d}}m_1(1+\sqrt{2})\mathfrak{B}}{(\min\{1,\varepsilon\})^{(2+\delta)}}\left[\sup_{n\in\mathbb{N}}\left(\frac{(n+1)^{\beta/n}\big((1+\sqrt{2})\mathfrak{B}(1+2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)\big)^{(2+\delta)}}{(m_n)^{\delta/2}}\right)^{n}\right] \qquad (4.50)
\end{aligned}$$

$$\cdot \Big(\max\Big\{1, \mathfrak{m}_{\mathfrak{p}} L 2^{\max\{q,1\}} d^{p+q}\big((1+L^2)^{q/2} + (q\mathfrak{p}+1)^{1/\mathfrak{p}}\big)\exp\big(\tfrac{[q(q\mathfrak{p}+3)+1]T}{2} + (L+1)T\big)\Big\}\Big)^{(2+\delta)}$$

$$= \frac{\alpha m_1 \mathfrak{B} d^{\mathfrak{d}+(p+q)(2+\delta)}}{(1+\sqrt{2})^{-1}(\min\{1,\varepsilon\})^{(2+\delta)}} \left[\sup_{n\in\mathbb{N}} \left(\frac{(n+1)^{\beta/n}\big((1+\sqrt{2})\mathfrak{B}(1+2LT)\exp\big((m_n)^{\mathfrak{p}/2}/n\big)\big)^{(2+\delta)}}{(m_n)^{\delta/2}}\right)^n\right]$$

$$\cdot \Big(\max\Big\{1, \mathfrak{m}_{\mathfrak{p}} L 2^{\max\{q,1\}}\big((1+L^2)^{q/2} + (q\mathfrak{p}+1)^{1/\mathfrak{p}}\big)\exp\big(\tfrac{[q(q\mathfrak{p}+3)+1]T}{2} + (L+1)T\big)\Big\}\Big)^{(2+\delta)} < \infty.$$

Combining (4.44) and (4.50) hence establishes (4.30). The proof of Proposition 4.4 is thus complete. □

4.4 Complexity analysis in the case of semilinear partial differential equations

Lemma 4.5. *Let $p \in (0,\infty)$ and let $\phi\colon \mathbb{R} \to \mathbb{N}$ satisfy for all $x \in [1,\infty)$ that $\phi(x) = \max\{k \in \mathbb{N}\colon k \le \exp(|\ln(x)|^{1/2})\}$. Then*

(i) it holds that $\limsup_{x\to\infty}\big[\frac{(\phi(x))^p}{x} + \frac{1}{\phi(x)}\big] = 0$ and

(ii) it holds for all $x \in \mathbb{N}$ that $\phi(x+1) \le 2\phi(x)$.

Proof of Lemma 4.5. Throughout this proof let $\psi\colon \mathbb{R} \to \mathbb{R}$ satisfy for all $x \in [1,\infty)$ that $\psi(x) = \exp(|\ln(x)|^{1/2})$. Note that the fact that for all $x \in [1,\infty)$ it holds that $\ln(x) \in [0,\infty)$ assures that for all $x \in (1,\infty)$ it holds that

$$\tfrac{d}{dx}(\psi(x))^p = \frac{p\big(\exp(|\ln(x)|^{1/2})\big)^p}{2x|\ln(x)|^{1/2}}. \tag{4.51}$$

This and the fact that for all $x \in [1,\infty)$ it holds that $\ln(x) \in [0,\infty)$ ensure that for all $x \in (1,\infty)$ it holds that

$$\tfrac{d^2}{dx^2}(\psi(x))^p = -\frac{p\big(\exp(|\ln(x)|^{1/2})\big)^p\big[2\ln(x) - p|\ln(x)|^{1/2} + 1\big]}{4x^2|\ln(x)|^{3/2}}. \tag{4.52}$$

Combining this and (4.51) shows that $(\exp([p+\sqrt{\max\{0,p^2-8\}}]/4),\infty) \ni x \mapsto \frac{d}{dx}(\psi(x))^p \in \mathbb{R}$ is decreasing. This, the fact (4.51) implies that for all $x \in (1,\infty)$ it holds that $\frac{d}{dx}(\psi(x))^p \in [0,\infty)$, and L'Hôpital's rule establish that

$$\limsup_{x\to\infty} \tfrac{(\psi(x))^p}{x} = \lim_{x\to\infty} \tfrac{(\psi(x))^p}{x} = \lim_{x\to\infty} \tfrac{d}{dx}(\psi(x))^p = 0. \tag{4.53}$$

Combining this, the fact that for all $x \in [1,\infty)$ it holds that $\phi(x) \le \psi(x)$, the fact that for all $x \in [1,\infty)$ it holds that $\phi(x) \in \mathbb{N}$, and the fact that for all $x \in [1,\infty)$ it holds that ϕ is non-decreasing proves that

$$\limsup_{x\to\infty}\big[\tfrac{(\phi(x))^p}{x} + \tfrac{1}{\phi(x)}\big] \le \limsup_{x\to\infty} \tfrac{(\phi(x))^p}{x} + \limsup_{x\to\infty} \tfrac{1}{\phi(x)} \le \limsup_{x\to\infty} \tfrac{(\psi(x))^p}{x} + \limsup_{x\to\infty} \tfrac{1}{\phi(x)} = 0. \tag{4.54}$$

This establishes item (i). Next note that for all $x \in (1,\infty)$ it holds that

$$\frac{\phi(x+1)}{\phi(x)} = \frac{\max\{k \in \mathbb{N}\colon k \le \exp(|\ln(1+x)|^{1/2})\}}{\max\{k \in \mathbb{N}\colon k \le \exp(|\ln(x)|^{1/2})\}} \le \frac{\exp(|\ln(1+x)|^{1/2})}{\exp(|\ln(x)|^{1/2}) - 1}. \tag{4.55}$$

In addition, observe that the fact that for all $x \in (1,\infty)$ it holds that $\ln(x) \in (0,\infty)$ demonstrates that for all $x \in (1,\infty)$ it holds that

$$\begin{aligned} &\frac{d}{dx}\frac{\exp((\ln(1+x))^{1/2})}{\exp((\ln(x))^{1/2}) - 1} \\ &= \frac{\exp(|\ln(x+1)|^{1/2})}{2[\exp(|\ln(x)|^{1/2}) - 1]}\left[\frac{1}{(1+x)|\ln(x+1)|^{1/2}} - \frac{\exp(|\ln(x)|^{1/2})}{x[\exp(|\ln(x)|^{1/2}) - 1]|\ln(x)|^{1/2}}\right] \le 0. \end{aligned} \tag{4.56}$$

This implies that for all $x \in [3, \infty)$ it holds that

$$\frac{\exp(|\ln(1+x)|^{1/2})}{\exp(|\ln(x)|^{1/2}) - 1} \leq \frac{\exp(|\ln(4)|^{1/2})}{\exp(|\ln(3)|^{1/2}) - 1} \leq 2. \tag{4.57}$$

Combining this and the fact that

$$\frac{\phi(2)}{\phi(1)} = \frac{\max\{k \in \mathbb{N} \colon k \leq \exp(|\ln(2)|^{1/2})\}}{\max\{k \in \mathbb{N} \colon k \leq \exp(|\ln(1)|^{1/2})\}} \leq \exp(|\ln(2)|^{1/2}) \leq \exp(\ln(2)) = 2 \tag{4.58}$$

proves that for all $x \in \mathbb{N}$ it holds that $\phi(x+1) \leq 2\phi(x)$. This establishes item (ii). The proof of Lemma 4.5 is thus complete. $\square$

Theorem 4.6. *Let $p, q, T, \kappa, \mathfrak{d} \in [0, \infty)$, $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$, $f \in C(\mathbb{R}, \mathbb{R})$, let $u_d \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, assume for all $d \in \mathbb{N}$, $t \in [0,T]$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ that $|f(v) - f(w)| \leq \kappa |v - w|$, $|u_d(t,x)| \leq \kappa d^p (1 + \sum_{k=1}^d |x_k|)^q$, and*

$$(\tfrac{\partial}{\partial t} u_d)(t,x) + (\Delta_x u_d)(t,x) + f(u_d(t,x)) = 0, \tag{4.59}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathfrak{u}^\theta \colon \Omega \to [0,1]$, $\theta \in \Theta$, be i.i.d. random variables, assume for all $r \in (0,1)$ that $\mathbb{P}(\mathfrak{u}^0 \leq r) = r$, let $\mathcal{U}^\theta \colon [0,T] \times \Omega \to [0,T]$, $\theta \in \Theta$, satisfy for all $t \in [0,T]$, $\theta \in \Theta$ that $\mathcal{U}^\theta_t = t + (T-t)\mathfrak{u}^\theta$, let $W^{d,\theta} \colon [0,T] \times \Omega \to \mathbb{R}^d$, $\theta \in \Theta$, $d \in \mathbb{N}$, be independent standard Brownian motions, assume for all $d \in \mathbb{N}$ that $(\mathcal{U}^\theta)_{\theta \in \Theta}$ and $(W^{d,\theta})_{\theta \in \Theta}$ are independent, let $\phi \colon \mathbb{N} \to \mathbb{N}$ and $U^{d,\theta}_{n,m} \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $d, n, m \in \mathbb{Z}$, $\theta \in \Theta$, satisfy for all $n \in \mathbb{N}_0$, $d, m \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that $\phi(m) = \max\{k \in \mathbb{N} \colon k \leq \exp(|\ln(m)|^{1/2})\}$ and

$$\begin{aligned} U^{d,\theta}_{n,m}(t,x) = \sum_{i=0}^{n-1} \tfrac{(T-t)}{(\phi(m))^{n-i}} \Bigg[\sum_{k=1}^{(\phi(m))^{n-i}} \Big[f\big(U^{d,(\theta,i,k)}_{i,m}\big(\mathcal{U}^{(\theta,i,k)}_t, x + \sqrt{2}\, W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}\big)\big) \\ - \mathbb{1}_{\mathbb{N}}(i)\, f\big(U^{d,(\theta,-i,k)}_{i-1,m}\big(\mathcal{U}^{(\theta,i,k)}_t, x + \sqrt{2}\, W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}\big)\big)\Big]\Bigg] + \tfrac{\mathbb{1}_{\mathbb{N}}(n)}{(\phi(m))^n} \Bigg[\sum_{k=1}^{(\phi(m))^n} u_d\big(T, x + \sqrt{2}\, W^{d,(\theta,0,-k)}_{T-t}\big)\Bigg], \end{aligned} \tag{4.60}$$

and let $\mathfrak{C}_{d,n,m} \in \mathbb{R}$, $d, n, m \in \mathbb{N}_0$, satisfy for all $d, m \in \mathbb{N}$, $n \in \mathbb{N}_0$ that

$$\mathfrak{C}_{d,n,m} \leq \mathbb{1}_{\mathbb{N}}(n)\, \kappa d^{\mathfrak{d}} (\phi(m))^n + \sum_{k=0}^{n-1} (\phi(m))^{n-k} \big(\kappa d^{\mathfrak{d}} + \mathfrak{C}_{d,k,m} + \mathfrak{C}_{d,\max\{k-1,0\},m}\big). \tag{4.61}$$

Then there exist $\mathcal{n} \colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ and $c = (c_{\mathfrak{p},\delta})_{(\mathfrak{p},\delta) \in \mathbb{R}^2} \colon \mathbb{R}^2 \to \mathbb{R}$ such that for all $d \in \mathbb{N}$, $\varepsilon, \delta, \mathfrak{p} \in (0, \infty)$ it holds that $\mathfrak{C}_{d,\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)} \leq c_{\mathfrak{p},\delta} d^{\mathfrak{d} + (p+q)(2+\delta)} (\min\{1, \varepsilon\})^{-(2+\delta)}$ and

$$\sup_{t \in [0,T]} \sup_{x \in [-\sqrt{2}\kappa, \sqrt{2}\kappa]^d} \Big(\mathbb{E}\Big[\big| u_d(t,x) - U^{d,0}_{\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)}(t,x) \big|^{\mathfrak{p}} \Big] \Big)^{1/\mathfrak{p}} \leq \varepsilon. \tag{4.62}$$

Proof of Theorem 4.6. Throughout this proof let $\varphi \colon (0,\infty) \to [2,\infty)$ satisfy for all $z \in (0,\infty)$ that $\varphi(z) = \max\{2, z\}$, let $\mathfrak{D} = \max\{2^{q/2}\kappa, |f(0)|\}$, let $\mathbb{F}^d_t \subseteq \mathcal{F}$, $d \in \mathbb{N}$, $t \in [0,T]$, satisfy for all $d \in \mathbb{N}$, $t \in [0,T]$ that

$$\mathbb{F}^d_t = \begin{cases} \bigcap_{s \in (t,T]} \sigma\big(\sigma(W^{d,0}_r \colon r \in [0,s]) \cup \{A \in \mathcal{F} \colon \mathbb{P}(A) = 0\}\big) & \colon t < T \\ \sigma\big(\sigma(W^{d,0}_s \colon s \in [0,T]) \cup \{A \in \mathcal{F} \colon \mathbb{P}(A) = 0\}\big) & \colon t = T \end{cases}, \tag{4.63}$$

let $a_d \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $d \in \mathbb{N}$, and $b_d \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^{d \times d})$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $t \in [0,T]$, $x, v \in \mathbb{R}^d$ that $a_d(t,x) = 0$ and $b_d(t,x)v = \sqrt{2}\, v$, let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots \in \mathbb{N}$ satisfy for all $j \in \mathbb{N}$ that $\mathcal{M}_j = \phi(j)$, let $V_d \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that $V_d(t,x) = u_d(t, \sqrt{2}\, x)$, let $F_d \colon [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$,

$w \in \mathbb{R}$ that $F_d(t,x,w) = f(w)$, and let $\mathfrak{V}^{d,\theta}_{n,j} \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $d,n,j \in \mathbb{Z}$, $\theta \in \Theta$, satisfy for all $d,n,j \in \mathbb{Z}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that $\mathfrak{V}^{d,\theta}_{n,j}(t,x) = U^{d,\theta}_{n,j}(t, \sqrt{2}\,x)$. Note that the hypothesis that for all $v,w \in \mathbb{R}$ it holds that $|f(v)-f(w)| \leq \kappa|v-w|$ assures that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $v,w \in \mathbb{R}$ it holds that

$$|F_d(t,x,v) - F_d(t,x,w)| = |f(v)-f(w)| \leq \kappa|v-w|. \tag{4.64}$$

Next observe that the hypothesis that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $|u_d(t,x)| \leq \kappa d^p (1+\sum_{k=1}^d |x_k|)^p$ ensures that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x = (x_1,x_2,\dots,x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &\max\{|F_d(t,x,0)|, |V_d(t,x)|\} = \max\{|f(0)|, |u_d(t,\sqrt{2}\,x)|\} \\ &\leq \max\{|f(0)|, \kappa d^p (1+\textstyle\sum_{k=1}^d |\sqrt{2}\,x_k|)^q\} \leq \mathfrak{D} d^p (1+\textstyle\sum_{k=1}^d |x_k|)^q. \end{aligned} \tag{4.65}$$

In addition, note that (4.63) guarantees that $\mathbb{F}^d_t \subseteq \mathcal{F}$, $d \in \mathbb{N}$, $t \in [0,T]$, satisfies that

(I) it holds for all $d \in \mathbb{N}$ that $\{A \in \mathcal{F} \colon \mathbb{P}(A) = 0\} \subseteq \mathbb{F}^d_0$ and

(II) it holds for all $d \in \mathbb{N}$, $t \in [0,T]$ that $\mathbb{F}^d_t = \cap_{s \in (t,T]} \mathbb{F}^d_s$.

Combining items (I) and (II), (4.63), and Hutzenthaler et al. [31, Lemma 2.17] (applied with $m \curvearrowleft d$, $T \curvearrowleft T$, $W \curvearrowleft W$, $\mathbb{H}_t \curvearrowleft \mathbb{F}^d_t$, $(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]}) \curvearrowleft (\Omega,\mathcal{F},\mathbb{P},(\sigma(W^{d,0}_s \colon s \in [0,t]) \cup \{A \in \mathcal{F} \colon \mathbb{P}(A)=0\})_{t\in[0,T]})$ in the notation of [31, Lemma 2.17]) therefore assures that for all $d \in \mathbb{N}$ it holds that $W^{d,0} \colon [0,T] \times \Omega \to \mathbb{R}^d$ is a standard $(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}^d_t)_{t\in[0,T]})$-Brownian motion. This, the hypothesis that for all $t \in [0,T]$, $x \in \mathbb{R}^d$, $v,w \in \mathbb{R}$ it holds that $|f(v)-f(w)| \leq \kappa|v-w|$, the hypothesis that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $|u_d(t,x)| \leq \kappa d^p(1+\sum_{k=1}^d |x_k|)^q$, and, e.g., Beck et al. [4, Corollary 3.9] (applied for every $d \in \mathbb{N}$ with $d \curvearrowleft d$, $m \curvearrowleft d$, $T \curvearrowleft T$, $L \curvearrowleft \max\{\sqrt{2d},\kappa\}$, $\mathfrak{C} \curvearrowleft 0$, $f \curvearrowleft f$, $g \curvearrowleft (\mathbb{R}^d \ni x \mapsto u_d(T,x) \in \mathbb{R})$, $\mu \curvearrowleft a_d$, $\sigma \curvearrowleft b_d$, $W^{d,0} \curvearrowleft W^{d,0}$, $(\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]}) \curvearrowleft (\Omega,\mathcal{F},\mathbb{P},(\mathbb{F}^d_t)_{t\in[0,T]})$ in the notation of [4, Corollary 3.9]) ensure that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $\mathbb{E}[|u_d(T,x+\sqrt{2}\,W^{d,0}_{T-t})| + \int_t^T |f(u_d(s,x+\sqrt{2}\,W^{d,0}_{s-t}))|\,ds] < \infty$ and

$$u_d(t,x) = \mathbb{E}\big[u_d(T,x+\sqrt{2}\,W^{d,0}_{T-t})\big] + \int_t^T \mathbb{E}\big[f(u_d(s,x+\sqrt{2}\,W^{d,0}_{s-t}))\big]\,ds. \tag{4.66}$$

Combining this, the fact that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $V_d(t,x) = u_d(t,\sqrt{2}\,x)$, and the fact that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $w \in \mathbb{R}$ it holds that $F_d(t,x,w) = f(w)$ demonstrates that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &\mathbb{E}\bigg[|V_d(T,x+W^{d,0}_{T-t})| + \int_t^T |F_d(s,x+W^{d,0}_{s-t},V_d(s,x+W^{d,0}_{s-t}))|\,ds\bigg] \\ &= \mathbb{E}\bigg[|V_d(T,x+W^{d,0}_{T-t})| + \int_t^T |f(V_d(s,x+W^{d,0}_{s-t}))|\,ds\bigg] \\ &= \mathbb{E}\bigg[|u_d(T,\sqrt{2}(x+W^{d,0}_{T-t}))| + \int_t^T |f(u_d(s,\sqrt{2}(x+W^{d,0}_{s-t})))|\,ds\bigg] < \infty \end{aligned} \tag{4.67}$$

and

$$\begin{aligned} V_d(t,x) = u_d(t,\sqrt{2}\,x) &= \mathbb{E}\big[u_d(T,\sqrt{2}(x+W^{d,0}_{T-t}))\big] + \int_t^T \mathbb{E}\big[f(u_d(s,\sqrt{2}(x+W^{d,0}_{s-t})))\big]\,ds \\ &= \mathbb{E}\big[V_d(T,x+W^{d,0}_{T-t})\big] + \int_t^T \mathbb{E}\big[f(V_d(s,x+W^{d,0}_{s-t}))\big]\,ds \\ &= \mathbb{E}\big[V_d(T,x+W^{d,0}_{T-t})\big] + \int_t^T \mathbb{E}\big[F_d(s,x+W^{d,0}_{s-t},V_d(s,x+W^{d,0}_{s-t}))\big]\,ds. \end{aligned} \tag{4.68}$$

Moreover, note that the fact that for all $j \in \mathbb{N}$ it holds that $\mathcal{M}_j = \phi(j)$ and (4.60) show that for all $n \in \mathbb{N}_0$, $d, j \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} U^{d,\theta}_{n,j}(t, \sqrt{2}\,x) = \sum_{i=0}^{n-1} \tfrac{(T-t)}{(\mathcal{M}_j)^{n-i}} \Bigg[\sum_{k=1}^{(\mathcal{M}_j)^{n-i}} \Big[f\big(U^{d,(\theta,i,k)}_{i,j}\big(\mathcal{U}^{(\theta,i,k)}_t, \sqrt{2}(x + W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t})\big)\big) \\ - \mathbb{1}_{\mathbb{N}}(i)\, f\big(U^{d,(\theta,-i,k)}_{i-1,j}\big(\mathcal{U}^{(\theta,i,k)}_t, \sqrt{2}(x + W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t})\big)\big)\Big]\Bigg] + \Bigg[\sum_{k=1}^{(\mathcal{M}_j)^n} \tfrac{\mathbb{1}_{\mathbb{N}}(n)\, u_d(T, \sqrt{2}(x + W^{d,(\theta,0,-k)}_{T-t}))}{(\mathcal{M}_j)^n} \Bigg]. \end{aligned} \tag{4.69}$$

This and the fact that for all $d, n, j \in \mathbb{Z}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $\mathfrak{V}^{d,\theta}_{n,j}(t,x) = U^{d,\theta}_{n,j}(t, \sqrt{2}\,x)$ imply that for all $n \in \mathbb{N}_0$, $d, j \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} U^{d,\theta}_{n,j}(t, \sqrt{2}\,x) &= \mathfrak{V}^{d,\theta}_{n,j}(t,x) \\ &= \sum_{i=0}^{n-1} \tfrac{(T-t)}{(\mathcal{M}_j)^{n-i}} \Bigg[\sum_{k=1}^{(\mathcal{M}_j)^{n-i}} \Big[f\big(\mathfrak{V}^{d,(\theta,i,k)}_{i,j}\big(\mathcal{U}^{(\theta,i,k)}_t, x + W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}\big)\big) \\ &\quad - \mathbb{1}_{\mathbb{N}}(i)\, f\big(\mathfrak{V}^{d,(\theta,-i,k)}_{i-1,j}\big(\mathcal{U}^{(\theta,i,k)}_t, x + W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}\big)\big)\Big]\Bigg] + \Bigg[\sum_{k=1}^{(\mathcal{M}_j)^n} \tfrac{\mathbb{1}_{\mathbb{N}}(n)\, V_d(T, x + W^{d,(\theta,0,-k)}_{T-t})}{(\mathcal{M}_j)^n} \Bigg]. \end{aligned} \tag{4.70}$$

Combining this and the fact that for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $w \in \mathbb{R}$ it holds that $F_d(t,x,w) = f(w)$ yields that for all $n \in \mathbb{N}_0$, $d, j \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \mathfrak{V}^{d,\theta}_{n,j}(t,x) &= \frac{\mathbb{1}_{\mathbb{N}}(n)}{(\mathcal{M}_j)^n} \Bigg[\sum_{k=1}^{(\mathcal{M}_j)^n} V_d\big(T, x + W^{d,(\theta,0,-k)}_{T-t}\big) \Bigg] \\ &\quad + \sum_{i=0}^{n-1} \frac{(T-t)}{(\mathcal{M}_j)^{n-i}} \Bigg[\sum_{k=1}^{(\mathcal{M}_j)^{n-i}} \Big[F_d\big(\mathcal{U}^{(\theta,i,k)}_t, x + W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}, \mathfrak{V}^{d,(\theta,i,k)}_{i,j}\big(\mathcal{U}^{(\theta,i,k)}_t, x + W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}\big)\big) \\ &\quad - \mathbb{1}_{\mathbb{N}}(i)\, F_d\big(\mathcal{U}^{(\theta,i,k)}_t, x + W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}, \mathfrak{V}^{d,(\theta,-i,k)}_{i-1,j}\big(\mathcal{U}^{(\theta,i,k)}_t, x + W^{d,(\theta,i,k)}_{\mathcal{U}^{(\theta,i,k)}_t - t}\big)\big)\Big]\Bigg]. \end{aligned} \tag{4.71}$$

Furthermore, observe that Lemma 4.5 and the fact that for all $m \in \mathbb{N}$ it holds that $\phi(m) = \max\{k \in \mathbb{N} \colon k \leq \exp(|\ln(m)|^{1/2})\}$ imply that

(A) it holds for all $\mathfrak{p} \in (0,\infty)$ that $\limsup_{j\to\infty}[{}^{(\mathcal{M}_j)^{m(\mathfrak{p})/2}}/_j + {}^1/_{\mathcal{M}_j}] = \limsup_{j\to\infty}[{}^{(\phi(j))^{m(\mathfrak{p})/2}}/_j + {}^1/_{\phi(j)}] = 0$ and

(B) it holds for all $j \in \mathbb{N}$ that $\mathcal{M}_{j+1} = \phi(j+1) \leq 2\phi(j) = 2\mathcal{M}_j$.

Combining items (A) and (B), (4.64), (4.65), (4.67), (4.68), (4.71), and Proposition 4.4 (applied with $\mathfrak{a} \curvearrowleft \kappa$, $\mathfrak{d} \curvearrowleft \mathfrak{d}$, $\alpha \curvearrowleft \kappa$, $\beta \curvearrowleft 0$, $p \curvearrowleft p$, $q \curvearrowleft q$, $\mathfrak{B} \curvearrowleft 2$, $L \curvearrowleft \mathfrak{D}$, $T \curvearrowleft T$, $(m_j)_{j\in\mathbb{N}} \curvearrowleft (\mathcal{M}_j)_{j\in\mathbb{N}}$, $\phi \curvearrowleft \phi$, $f_d \curvearrowleft F_d$, $g_d \curvearrowleft (\mathbb{R}^d \ni x \mapsto V_d(T,x) \in \mathbb{R})$, $u_d \curvearrowleft V_d$, $(\Omega, \mathcal{F}, \mathbb{P}) \curvearrowleft (\Omega, \mathcal{F}, \mathbb{P})$, $\mathfrak{u}^\theta \curvearrowleft \mathfrak{u}^\theta$, $\mathcal{U}^\theta \curvearrowleft \mathcal{U}^\theta$, $W^{d,\theta} \curvearrowleft W^{d,\theta}$, $U^{d,\theta}_{n,j} \curvearrowleft \mathfrak{V}^{d,\theta}_{n,j}$ in the notation of Proposition 4.4) hence guarantees that there exists $\mathcal{n} \colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ and $c = (c_{\mathfrak{p},\delta})_{(\mathfrak{p},\delta)\in\mathbb{R}^2} \colon \mathbb{R}^2 \to \mathbb{R}$ such that for all $d \in \mathbb{N}$, $\varepsilon, \delta \in (0,\infty)$, $\mathfrak{p} \in [2,\infty)$ it holds that $\mathfrak{C}_{d,\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)} \leq c_{\mathfrak{p},\delta} d^{\mathfrak{d}+(p+q)(2+\delta)} (\min\{1,\varepsilon\})^{-(2+\delta)}$ and

$$\sup_{t\in[0,T]} \sup_{x\in[-\kappa,\kappa]^d} \big(\mathbb{E}\big[|V_d(t,x) - \mathfrak{V}^{d,0}_{\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)}(t,x)|^{\mathfrak{p}}\big]\big)^{1/\mathfrak{p}} \leq \varepsilon. \tag{4.72}$$

This and Hölder's inequality prove that there exist $\mathcal{n} \colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ and $c = (c_{\mathfrak{p},\delta})_{(\mathfrak{p},\delta)\in\mathbb{R}^2} \colon \mathbb{R}^2 \to \mathbb{R}$ such that for all $d \in \mathbb{N}$, $\varepsilon, \delta, \mathfrak{p} \in (0,\infty)$ it holds that

$$\mathfrak{C}_{d,\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)} \leq c_{\mathfrak{p},\delta} d^{\mathfrak{d}+(p+q)(2+\delta)} (\min\{1,\varepsilon\})^{-(2+\delta)} \tag{4.73}$$

and

$$
\begin{aligned}
&\sup_{t\in[0,T]}\sup_{x\in[-\kappa,\kappa]^d}\mathbb{E}\big[|V_d(t,x)-\mathfrak{V}^{d,0}_{\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)}(t,x)|^{\mathfrak{p}}\big]\\
&\le \sup_{t\in[0,T]}\sup_{x\in[-\kappa,\kappa]^d}\big(\mathbb{E}\big[|V_d(t,x)-\mathfrak{V}^{d,0}_{\mathcal{n}(d,\varepsilon),\mathcal{n}(d,\varepsilon)}(t,x)|^{\varphi(\mathfrak{p})}\big]\big)^{\mathfrak{p}/\varphi(\mathfrak{p})}\le\varepsilon^{\mathfrak{p}}.
\end{aligned}
\tag{4.74}
$$

Combining (4.73), (4.74), and the fact that for all $n\in\mathbb{N}_0$, $d,j\in\mathbb{N}$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ it holds that $u_d(t,x)=V_d(t,{}^{x}\!/\!\sqrt{2})$ and $U^{d,\theta}_{n,j}(t,x)=\mathfrak{V}^{d,\theta}_{n,j}(t,{}^{x}\!/\!\sqrt{2})$ hence establishes (4.62). The proof of Theorem 4.6 is thus complete. □

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