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Regularity Properties for Solutions of Infinite Dimensional Kolmogorov Equations in Hilbert Spaces

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Abstract In this article we establish regularity properties for solutions of infinite dimensional Kolmogorov equations. We prove that if the nonlinear drift coefficients, the nonlinear diffusion coefficients, and the initial conditions of the considered Kolmogorov equations are n-times continuously Fréchet differentiable, then so are the generalized solutions at every positive time. In addition, a key contribution of this work is to prove suitable enhanced regularity properties for the derivatives of the generalized solutions of the Kolmogorov equations in the sense that the dominating linear operator in the drift coefficient of the Kolmogorov equation regularizes the higher order derivatives of the solutions. Such enhanced regularity properties are of major importance for establishing weak convergence rates for spatial and temporal numerical approximations of stochastic partial differential equations.

Keywords Infinite dimensional Kolmogorov equations · Generalized solutions

1 Introduction

In this article we establish regularity properties for solutions of infinite dimensional Kolmogorov equations. Infinite dimensional Kolmogorov equations are the Kolmogorov

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equations associated to stochastic partial differential equations (SPDEs) and such equations have been intensively studied in the literature in the last three decades (cf., e.g., Ma & Röckner [19], Röckner [21], Zabczyk [29], Cerrai [6], Da Prato & Zabczyk [12], Röckner & Sobol [23], Da Prato [9], Röckner [22], Röckner & Sobol [24], Röckner & Sobol [25], Da Prato [10], and the references mentioned therein). In Theorem 1.1 below we summarize some of the main findings of this paper. In our formulation of Theorem 1.1 we employ the following notation. For every $n \in \mathbb{N} = \{1, 2, \dots, \}$ and every non-trivial $\mathbb{R}$-Banach space $(V, \|\cdot\|_V)$ we denote by $C_b^n(V, \mathbb{R})$ the set of all n-times continuously Fréchet differentiable functions $f : V \to \mathbb{R}$ with globally bounded derivatives, we denote by $\|\cdot\|_{C_b^n(V,\mathbb{R})}$ the associated norm on $C_b^n(V, \mathbb{R})$ (cf. (6) below), we denote by $\mathrm{Lip}^n(V, \mathbb{R})$ the set of all functions $f : V \to \mathbb{R}$ in $C_b^n(V, \mathbb{R})$ which have globally Lipschitz continuous derivatives, and we denote by $|\cdot|_{\mathrm{Lip}^n(V,\mathbb{R})}$ an associated semi-norm on $\mathrm{Lip}^n(V, \mathbb{R})$ (cf. (7) below).

Theorem 1.1 *Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ and $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ be non-trivial separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $T \in (0, \infty)$, $\eta \in \mathbb{R}$, $n \in \mathbb{N}$, $F \in C_b^n(H, H)$, $B \in C_b^n(H, HS(U, H))$, and let $A : D(A) \subseteq H \to H$ be a generator of a strongly continuous analytic semigroup with* spectrum$(A) \subseteq \{z \in \mathbb{C} : \mathrm{Re}(z) < \eta\}$. *Then*

(i) *it holds that there exist unique functions $P_t : C_b^1(H, \mathbb{R}) \to C(H, \mathbb{R})$, $t \in [0, T]$, such that for every $\varphi \in C_b^1(H, \mathbb{R})$ it holds that $(P_t\varphi)(x) \in \mathbb{R}$, $(t, x) \in [0, T] \times H$, is a generalized solution*[1] *of*

$$\tfrac{\partial}{\partial t}(P_t\varphi)(x) = \tfrac{1}{2} \sum_{u\in\mathbb{U}} (P_t\varphi)''(x)(B(x)u, B(x)u) + (P_t\varphi)'(x)[Ax + F(x)] \quad (1)$$

for $(t, x) \in (0, T] \times D(A)$ with $(P_0\varphi)(x) = \varphi(x)$ for $x \in H$,

(ii) *it holds for all $k \in \{1, \dots, n\}$, $t \in [0, T]$ that $P_t(C_b^k(H, \mathbb{R})) \subseteq C_b^k(H, \mathbb{R})$,*

(iii) *it holds for all $k \in \{1, \dots, n\}$, $t \in [0, T]$ with $|F|_{\mathrm{Lip}^k(H,H)} + |B|_{\mathrm{Lip}^k(H,HS(U,H))} < \infty$ that $P_t(\mathrm{Lip}^k(H, \mathbb{R})) \subseteq \mathrm{Lip}^k(H, \mathbb{R})$,*

(iv) *it holds for all $k \in \{1, \dots, n\}$, $\delta_1, \dots, \delta_k \in [0, 1/2)$ with $\sum_{i=1}^k \delta_i < 1/2$ that*

$$\sup_{\varphi\in C_b^k(H,\mathbb{R})\setminus\{0\}} \sup_{x\in H} \sup_{u_1,\dots,u_k\in H\setminus\{0\}} \sup_{t\in(0,T]} \left[\frac{t^{\sum_{i=1}^k \delta_i} \, |(P_t\varphi)^{(k)}(x)(u_1, \dots, u_k)|}{\|\varphi\|_{C_b^k(H,\mathbb{R})} \prod_{i=1}^k \|(\eta - A)^{-\delta_i} u_i\|_H} \right] < \infty, \quad (2)$$

and

(v) *it holds for all $k \in \{1, \dots, n\}$, $\delta_1, \dots, \delta_k \in [0, 1/2)$ with $\sum_{i=1}^k \delta_i < 1/2$ and $|F|_{\mathrm{Lip}^k(H,H)} + |B|_{\mathrm{Lip}^k(H,HS(U,H))} < \infty$ that*

$$\sup_{\varphi\in \mathrm{Lip}^k(H,\mathbb{R})\setminus\{0\}} \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\substack{u_1,\dots,u_k\\ \in H\setminus\{0\}}} \sup_{t\in(0,T]} \left[\frac{t^{\sum_{i=1}^k \delta_i} |[(P_t\varphi)^{(k)}(x) - (P_t\varphi)^{(k)}(y)](u_1, \dots, u_k)|}{\|\varphi\|_{\mathrm{Lip}^k(H,\mathbb{R})} \|x - y\|_H \prod_{i=1}^k \|(\eta - A)^{-\delta_i} u_i\|_H} \right] < \infty. \quad (3)$$

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[1]The hypothesis that $(P_t\varphi)(x) \in \mathbb{R}$, $(t, x) \in [0, T] \times H$, is a generalized solution of (1) means that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every Id_U-cylindrical $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$-Wiener process $(W_t)_{t\in[0,T]}$, every $x \in H$, every continuous $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-adapted stochastic process $X^x : [0, T] \times \Omega \to H$ with

$$\forall t \in [0, T] : \mathbb{P}\Big(X_t^x = e^{tA}x + \int_0^t e^{(t-s)A} F(X_s^x)\, ds + \int_0^t e^{(t-s)A} B(X_s^x)\, dW_s \Big) = 1,$$

and every $t \in [0, T]$ it holds that $(P_t\varphi)(x) = \mathbb{E}[\varphi(X_t^x)]$ (cf., e.g., [11, Item (i) of Theorem 7.4] and [12, page 127]).

Note that Theorem 6.1 in Frieler & Knoche [14] establishes a quite similar result as item (ii) of Theorem 1.1 above in the specific case $n = 2$ (cf. also Theorem 6.7 in Zabczyk [29] and Theorem 7.4.3 in Da Prato & Zabczyk [12]). Theorem 1.1 above is a straightforward consequence of Theorem 3.3 in Section 3 below. In Theorem 3.3 below we also specify for every natural number $n \in \mathbb{N}$ and every $t \in [0, T]$ an explicit formula for the n-th derivative of the generalized solution $H \ni x \mapsto (P_t\varphi)(x) \in \mathbb{R}$ of (1) at time $t \in [0, T]$. Moreover, Theorem 3.3 below provides explicit bounds for the left hand sides of (2) and (3) (see items (vii) and (x) in Theorem 3.3 below). Next we would like to emphasize that Theorem 1.1 and Theorem 3.3, respectively, prove finiteness of (2) and (3) even though the denominators in (2) and (3) contain rather weak norms from negative Sobolev-type spaces for the multilinear arguments of the derivatives of the generalized solution. In particular, item (iv) in Theorem 1.1 above and item (vii) in Theorem 3.3 below, respectively, reveal for every $p \in [1, \infty)$, $k \in \{1, 2, \dots, n\}$, $\delta_1, \delta_2, \dots, \delta_k \in [0, 1/2)$, $x \in H$, $t \in (0, T]$ that the k-th derivative $(P_t\varphi)^{(k)}(x)$ even takes values in the continuously embedded subspace

$$L_k(H_{-\delta_1} \times H_{-\delta_2} \times \cdots \times H_{-\delta_k}, \mathbb{R}) \tag{4}$$

of $L_k(H^k, \mathbb{R}) = L_k(H \times H \times \cdots \times H, \mathbb{R})$ provided that the hypothesis

$$\textstyle\sum_{i=1}^{k} \delta_i < 1/2 \tag{5}$$

is satisfied. Here $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, is a family of interpolation spaces associated to $\eta - A$ (cf., e.g., [26, Section 3.7]) and here we denote for every $k \in \mathbb{N}$ and all $\mathbb{R}$-Banach spaces $(V_1, \|\cdot\|_{V_1}), \dots, (V_k, \|\cdot\|_{V_k})$ by $L_k(V_1 \times V_2 \times \cdots \times V_k, \mathbb{R})$ the $\mathbb{R}$-Banach space of all continuous k-linear functions from $V_1 \times V_2 \times \cdots \times V_k$ to $\mathbb{R}$. In addition, we employ items (iv)–(v) in Theorem 1.1 above and items (vii) and (x) in Theorem 3.3 below, respectively, to establish similar a priori bounds as (2)–(3) for a family of appropriately mollified solutions of (1) which hold uniformly in the mollification parameter; see items (iv)–(v) in Corollary 4.2 below for details. Items (iv)–(v) in Theorem 1.1 above, items (vii) and (x) in Theorem 3.3 below, and, especially, items (iv)–(v) in Corollary 4.2 below, respectively, are of major importance for establishing essentially sharp probabilistically *weak convergence rates* for numerical approximation processes as the analytically weak norms for the multilinear arguments of the derivatives of the generalized solution (cf. the denominators in (2) and (3) above) translate in analytically weak norms for the approximation errors in the probabilistically weak error analysis which, in turn, result in essentially sharp probabilistically weak convergence rates for the numerical approximation processes (cf., e.g., Theorem 2.2 in Debussche [13], Theorem 2.1 in Wang & Gan [28], Theorem 1.1 in Andersson & Larsson [3], Theorem 1.1 in Bréhier [4], Theorem 5.1 in Bréhier & Kopec [5], Corollary 1 in Wang [27], Corollary 5.2 in Conus et al. [8], Theorem 6.1 in Kopec [18], and Corollary 8.2 in [17]).

1.1 Notation

In this section we introduce some of the notation which we employ throughout the article (cf., e.g., [1, Section 1.1]). For two sets A and B we denote by $\mathbb{M}(A, B)$ the set of all mappings from A to B. For two measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A}, \mathcal{B})$ the set of $\mathcal{A}/\mathcal{B}$-measurable functions. For a set A we denote by $\mathcal{P}(A)$ the power set of A and we denote by $\#_A \in \mathbb{N}_0 \cup \{\infty\}$ the number of elements of A. For a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ we denote by $\mu_A \colon \mathcal{B}(A) \to [0, \infty]$ the Lebesgue-Borel measure on A. We denote by $\lfloor\cdot\rfloor \colon \mathbb{R} \to \mathbb{R}$ and $\lceil\cdot\rceil \colon \mathbb{R} \to \mathbb{R}$ the functions which satisfy for all $t \in \mathbb{R}$ that $\lfloor t \rfloor = \max((-\infty, t] \cap \{0, 1, -1, 2, -2, \dots\})$ and $\lceil t \rceil =$

$\min([t,\infty)\cap\{0,1,-1,2,-2,\dots\})$. For $\mathbb{R}$-Banach spaces $(V,\|\cdot\|_V)$ and $(W,\|\cdot\|_W)$ with $\#_V>1$ and a natural number $n\in\mathbb{N}$ we denote by $|\cdot|_{C_b^n(V,W)}\colon C^n(V,W)\to[0,\infty]$ and $\|\cdot\|_{C_b^n(V,W)}\colon C^n(V,W)\to[0,\infty]$ the functions which satisfy for all $f\in C^n(V,W)$ that

$$|f|_{C_b^n(V,W)}=\sup\nolimits_{x\in V}\left\|f^{(n)}(x)\right\|_{L^{(n)}(V,W)},\qquad \|f\|_{C_b^n(V,W)}=\|f(0)\|_W+\textstyle\sum_{k=1}^n|f|_{C_b^k(V,W)} \tag{6}$$

and we denote by $C_b^n(V,W)$ the set given by $C_b^n(V,W)=\{f\in C^n(V,W)\colon\ \|f\|_{C_b^n(V,W)}<\infty\}$. For $\mathbb{R}$-Banach spaces $(V,\|\cdot\|_V)$ and $(W,\|\cdot\|_W)$ with $\#_V>1$ and a nonnegative integer $n\in\mathbb{N}_0$ we denote by $|\cdot|_{\mathrm{Lip}^n(V,W)}\colon C^n(V,W)\to[0,\infty]$ and $\|\cdot\|_{\mathrm{Lip}^n(V,W)}\colon C^n(V,W)\to[0,\infty]$ the functions which satisfy for all $f\in C^n(V,W)$ that

$$|f|_{\mathrm{Lip}^n(V,W)}=\begin{cases}\sup\limits_{x,y\in V,\,x\neq y}\left(\dfrac{\|f(x)-f(y)\|_W}{\|x-y\|_V}\right) & : n=0\\[2ex] \sup\limits_{x,y\in V,\,x\neq y}\left(\dfrac{\|f^{(n)}(x)-f^{(n)}(y)\|_{L^{(n)}(V,W)}}{\|x-y\|_V}\right) & : n\in\mathbb{N}\end{cases}, \tag{7}$$
$$\|f\|_{\mathrm{Lip}^n(V,W)}=\|f(0)\|_W+\textstyle\sum_{k=0}^n|f|_{\mathrm{Lip}^k(V,W)}$$

and we denote by $\mathrm{Lip}^n(V,W)$ the set given by $\mathrm{Lip}^n(V,W)=\{f\in C^n(V,W)\colon\ \|f\|_{\mathrm{Lip}^n(V,W)}<\infty\}$. We denote by $\Pi_k,\Pi_k^*\in\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))$, $k\in\mathbb{N}_0$, the sets which satisfy for all $k\in\mathbb{N}$ that $\Pi_0=\Pi_0^*=\emptyset$, $\Pi_k^*=\Pi_k\backslash\{\{\{1,2,\dots,k\}\}\}$, and

$$\begin{aligned}\Pi_k\ =\ \big\{A\subseteq\mathcal{P}(\mathbb{N})\colon\ &[\emptyset\notin A]\wedge[\cup_{a\in A}a=\{1,2,\dots,k\}]\\ &\wedge\ [\forall\, a,b\in A\colon\ (a\neq b\Rightarrow a\cap b=\emptyset)]\big\}\end{aligned} \tag{8}$$

(see, e.g., (11) in Andersson et al. [2]). For a natural number $k\in\mathbb{N}$ and a set $\varpi\in\Pi_k$ we denote by $I_1^\varpi,I_2^\varpi,\dots,I_{\#_\varpi}^\varpi\in\varpi$ the sets which satisfy that $\min(I_1^\varpi)<\min(I_2^\varpi)<\dots<\min(I_{\#_\varpi}^\varpi)$. For a natural number $k\in\mathbb{N}$, a set $\varpi\in\Pi_k$, and a natural number $i\in\{1,2,\dots,\#_\varpi\}$ we denote by $I_{i,1}^\varpi,I_{i,2}^\varpi,\dots,I_{i,\#_{I_i^\varpi}}^\varpi\in I_i^\varpi$ the natural numbers which satisfy that $I_{i,1}^\varpi<I_{i,2}^\varpi<\dots<I_{i,\#_{I_i^\varpi}}^\varpi$. For a measure space $(\Omega,\mathcal{F},\mu)$, a measurable space $(S,\mathcal{S})$, a set R, and a function $f\colon\Omega\to R$ we denote by $[f]_{\mu,\mathcal{S}}$ the set given by

$$[f]_{\mu,\mathcal{S}}=\{g\in\mathcal{M}(\mathcal{F},\mathcal{S})\colon(\exists\, A\in\mathcal{F}\colon\ \mu(A)=0\text{ and }\{\omega\in\Omega\colon\ f(\omega)\neq g(\omega)\}\subseteq A)\}. \tag{9}$$

1.2 Setting

Throughout this article the following setting is frequently used. Let $T\in(0,\infty)$, $\eta\in\mathbb{R}$, let $(H,\|\cdot\|_H,\langle\cdot,\cdot\rangle_H)$ and $(U,\|\cdot\|_U,\langle\cdot,\cdot\rangle_U)$ be separable $\mathbb{R}$-Hilbert spaces with $\#_H>1$, let $(V,\|\cdot\|_V)$ be a separable $\mathbb{R}$-Banach space, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$-Wiener process, let $A\colon D(A)\subseteq H\to H$ be a generator of a strongly continuous analytic semigroup with $\mathrm{spectrum}(A)\subseteq\{z\in\mathbb{C}\colon\mathrm{Re}(z)<\eta\}$, let $(H_r,\|\cdot\|_{H_r},\langle\cdot,\cdot\rangle_{H_r})$, $r\in\mathbb{R}$, be a family of interpolation spaces associated to $\eta-A$, for every $k\in\mathbb{N}$, $\varpi\in\Pi_k$, $i\in\{1,2,\dots,\#_\varpi\}$ let $[\cdot]_i^\varpi\colon H^{k+1}\to H^{\#_{I_i^\varpi}+1}$ be the mapping which satisfies for all $\mathbf{u}=(u_0,u_1,\dots,u_k)\in H^{k+1}$ that $[\mathbf{u}]_i^\varpi=(u_0,u_{I_{i,1}^\varpi},u_{I_{i,2}^\varpi},\dots,u_{I_{i,\#_{I_i^\varpi}}^\varpi})$, for every $k\in\mathbb{N}$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\dots,\delta_k)\in\mathbb{R}^k$, $\alpha\in[0,1)$, $\beta\in[0,1/2)$, $J\in\mathcal{P}(\mathbb{R})$ let $\iota_J^{\boldsymbol{\delta},\alpha,\beta}\in\mathbb{R}$ be the real number given by $\iota_J^{\boldsymbol{\delta},\alpha,\beta}=\sum_{i\in J\cap\{1,2,\dots,k\}}\delta_i-\mathbb{1}_{[2,\infty)}(\#_{J\cap\{1,2,\dots,k\}})\min\{1-\alpha,1/2-\beta\}$, and for every separable $\mathbb{R}$-Banach space $(J,\|\cdot\|_J)$ and every $a\in\mathbb{R}$, $b\in(a,\infty)$, $I\in\mathcal{B}(\mathbb{R})$,

$X \in \mathcal{M}(\mathcal{B}(I) \otimes \mathcal{F}, \mathcal{B}(J))$ with $(a,b) \subseteq I$ let $\int_a^b X_s \, \mathbf{ds} \in L^0(\mathbb{P}; J)$ be the set given by $\int_a^b X_s \, \mathbf{ds} = \big[\int_a^b \mathbb{1}_{\{\int_a^b \|X_u\|_J \, du < \infty\}} X_s \, ds\big]_{\mathbb{P}, \mathcal{B}(J)}$.

2 Some Auxiliary Results for the Differentiation of Random Fields

Lemma 2.1 (A chain rule for random fields) *Let $(U, \|\cdot\|_U)$ be an $\mathbb{R}$-Banach space with $\#_U > 1$, let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be separable $\mathbb{R}$-Banach spaces, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X^{k,\mathbf{u}} \in \cap_{p \in [1,\infty)} \mathcal{L}^p(\mathbb{P}; V)$, $\mathbf{u} \in U^{k+1}$, $k \in \{0,1\}$, satisfy for all $p \in [1,\infty)$, $x, u \in U$ that $\big(U \ni y \mapsto [X^{0,y}]_{\mathbb{P},\mathcal{B}(V)} \in L^p(\mathbb{P}; V)\big) \in C^1(U, L^p(\mathbb{P}; V))$ and $\big(\frac{d}{dx}[X^{0,x}]_{\mathbb{P},\mathcal{B}(V)}\big)u = [X^{1,(x,u)}]_{\mathbb{P},\mathcal{B}(V)}$, and let $\varphi \in C^1(V, W)$ satisfy that $\limsup_{p \nearrow \infty} \sup_{x \in V} \frac{\|\varphi'(x)\|_{L(V,W)}}{|\max\{1, \|x\|_V\}|^p} < \infty$. Then*

(i) *it holds for all $x, u \in U$ that $\mathbb{E}[\|\varphi(X^{0,x})\|_W + \|\varphi'(X^{0,x}) X^{1,(x,u)}\|_W] < \infty$,*
(ii) *it holds that $\big(U \ni x \mapsto \mathbb{E}[\varphi(X^{0,x})] \in W\big) \in C^1(U, W)$, and*
(iii) *it holds for all $x, u \in U$ that $\big(\frac{d}{dx}\mathbb{E}[\varphi(X^{0,x})]\big)u = \mathbb{E}[\varphi'(X^{0,x}) X^{1,(x,u)}]$.*

Proof Throughout this proof let $c_{k,r} \in [0,\infty]$, $r \in (0,\infty)$, $k \in \{0,1\}$, be the extended real numbers which satisfy for all $r \in (0,\infty)$ that

$$c_{0,r} = \sup_{x \in V} \left[\frac{\|\varphi(x)\|_W}{|\max\{1, \|x\|_V\}|^r}\right] \qquad \text{and} \qquad c_{1,r} = \sup_{x \in V} \left[\frac{\|\varphi'(x)\|_{L(V,W)}}{|\max\{1, \|x\|_V\}|^r}\right] \tag{10}$$

and let $p \in [1,\infty)$ be a real number which satisfies that $c_{1,p} < \infty$. We note that the fundamental theorem of calculus implies that for all $x \in V$ it holds that

$$\begin{aligned} \|\varphi(x) - \varphi(0)\|_W &= \left\|\int_0^1 \varphi'(\rho x) x \, d\rho\right\|_W \leq \int_0^1 \|\varphi'(\rho x)\|_{L(V,W)} \, \|x\|_V \, d\rho \\ &\leq c_{1,p} \, \|x\|_V \sup\nolimits_{\rho \in [0,1]} |\max\{1, \|\rho x\|_V\}|^p = c_{1,p} \, \|x\|_V \, |\max\{1, \|x\|_V\}|^p \\ &\leq c_{1,p} \, |\max\{1, \|x\|_V\}|^{(p+1)}. \end{aligned} \tag{11}$$

This ensures that $c_{0,p+1} < \infty$. Hölder's inequality and the fact that $c_{1,p} < \infty$ therefore show that for all $x, u \in U$ it holds that

$$\begin{aligned} &\mathbb{E}[\|\varphi'(X^{0,x}) X^{1,(x,u)}\|_W] \leq c_{1,p} \, \mathbb{E}[|\max\{1, \|X^{0,x}\|_V\}|^p \, \|X^{1,(x,u)}\|_V] \\ &\leq c_{1,p} \, \|\max\{1, \|X^{0,x}\|_V\}\|^p_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \, \|X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} < \infty \end{aligned} \tag{12}$$

and

$$\mathbb{E}[\|\varphi(X^{0,x})\|_W] \leq c_{0,p+1} \, \mathbb{E}[|\max\{1, \|X^{0,x}\|_V\}|^{(p+1)}] < \infty. \tag{13}$$

This proves item (i). Next note that (12) and the fact that $\forall\, q \in [1,\infty),\ x \in U\colon \big(U \ni u \mapsto [X^{1,(x,u)}]_{\mathbb{P},\mathcal{B}(V)} \in L^q(\mathbb{P}; V)\big) \in L(U, L^q(\mathbb{P}; V))$ ensure that for every $x \in U$ it holds

a) that

$$\begin{aligned} &\sup_{u \in U,\, \|u\|_U = 1} \|\mathbb{E}[\varphi'(X^{0,x}) X^{1,(x,u)}]\|_W \\ &\leq c_{1,p} \, \|\max\{1, \|X^{0,x}\|_V\}\|^p_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \sup_{u \in U,\, \|u\|_U = 1} \|X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} < \infty \end{aligned} \tag{14}$$

and

b) that the function $\big(U \ni u \mapsto \mathbb{E}[\varphi'(X^{0,x}) X^{1,(x,u)}] \in W\big)$ is linear.

Hence, we obtain that

$$\big(U \ni u \mapsto \mathbb{E}[\varphi'(X^{0,x})X^{1,(x,u)}] \in W\big) \in L(U, W). \tag{15}$$

In the next step we demonstrate that for all $x \in U$ it holds that

$$\limsup_{U\setminus\{0\}\ni u \to 0} \left(\frac{\|\mathbb{E}[\varphi(X^{0,x+u})] - \mathbb{E}[\varphi(X^{0,x})] - \mathbb{E}[\varphi'(X^{0,x})X^{1,(x,u)}]\|_W}{\|u\|_U} \right) = 0. \tag{16}$$

For this we first observe that for all $x, u \in U$ it holds that

$$\begin{aligned} &\|\mathbb{E}[\varphi(X^{0,x+u})] - \mathbb{E}[\varphi(X^{0,x})] - \mathbb{E}[\varphi'(X^{0,x})X^{1,(x,u)}]\|_W \\ &\le \|\mathbb{E}[\varphi(X^{0,x+u}) - \varphi(X^{0,x}) - \varphi'(X^{0,x})(X^{0,x+u} - X^{0,x})]\|_W \\ &+ \|\mathbb{E}[\varphi'(X^{0,x})(X^{0,x+u} - X^{0,x} - X^{1,(x,u)})]\|_W. \end{aligned} \tag{17}$$

Moreover, we note that Hölder's inequality and the fact that $c_{1,p} < \infty$ ensure that for all $x \in U$ it holds that

$$\begin{aligned} &\limsup_{U\setminus\{0\}\ni u \to 0} \left(\frac{\|\mathbb{E}[\varphi'(X^{0,x})(X^{0,x+u} - X^{0,x} - X^{1,(x,u)})]\|_W}{\|u\|_U} \right) \\ &\le \|\varphi'(X^{0,x})\|_{\mathcal{L}^2(\mathbb{P};L(V,W))} \limsup_{U\setminus\{0\}\ni u \to 0} \left(\frac{\|X^{0,x+u} - X^{0,x} - X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)}}{\|u\|_U} \right) \\ &\le c_{1,p} \, \|\max\{1, \|X^{0,x}\|_V\}\|^p_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \limsup_{U\setminus\{0\}\ni u \to 0} \left(\frac{\|X^{0,x+u} - X^{0,x} - X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)}}{\|u\|_U} \right) = 0. \end{aligned} \tag{18}$$

Furthermore, we observe that the fundamental theorem of calculus shows that for all $x, u \in U$ it holds that

$$\begin{aligned} &\|\varphi(X^{0,x+u}) - \varphi(X^{0,x}) - \varphi'(X^{0,x})(X^{0,x+u} - X^{0,x})\|_W \\ &= \left\| \int_0^1 \big[\varphi'(X^{0,x} + \rho[X^{0,x+u} - X^{0,x}]) - \varphi'(X^{0,x})\big](X^{0,x+u} - X^{0,x}) \, d\rho \right\|_W \\ &\le \|X^{0,x+u} - X^{0,x}\|_V \int_0^1 \|\varphi'(X^{0,x} + \rho[X^{0,x+u} - X^{0,x}]) - \varphi'(X^{0,x})\|_{L(V,W)} \, d\rho. \end{aligned} \tag{19}$$

Hölder's inequality and Jensen's inequality therefore imply that for all $x, u \in U$ it holds that

$$\begin{aligned} &\|\mathbb{E}[\varphi(X^{0,x+u}) - \varphi(X^{0,x}) - \varphi'(X^{0,x})(X^{0,x+u} - X^{0,x})]\|_W \\ &\le \left\{ \mathbb{E}\left[\int_0^1 \|\varphi'(X^{0,x} + \rho[X^{0,x+u} - X^{0,x}]) - \varphi'(X^{0,x})\|^2_{L(V,W)} \, d\rho \right] \right\}^{1/2} \\ &\cdot \|X^{0,x+u} - X^{0,x}\|_{\mathcal{L}^2(\mathbb{P};V)}. \end{aligned} \tag{20}$$

Moreover, note that for all $q \in (2, \infty)$, $\rho \in [0, 1]$, $x, y \in U$ it holds that

$$\begin{aligned} &\mathbb{E}\big[\|\varphi'(X^{0,x} + \rho[X^{0,y} - X^{0,x}])\|^q_{L(V,W)}\big] \\ &\le |c_{1,p}|^q \, \mathbb{E}\big[|\max\{1, \|X^{0,x} + \rho[X^{0,y} - X^{0,x}]\|^p_V\}|^q\big] \\ &\le |c_{1,p}|^q \, \mathbb{E}\big[|\max\{1, \|X^{0,x}\|_V, \|X^{0,y}\|_V\}|^{pq}\big] \\ &\le |c_{1,p}|^q \big(1 + \mathbb{E}\big[\|X^{0,x}\|^{pq}_V\big] + \mathbb{E}\big[\|X^{0,y}\|^{pq}_V\big]\big). \end{aligned} \tag{21}$$

This and the fact that $\forall q \in [1,\infty)\colon \big(U \ni x \mapsto [X^{0,x}]_{\mathbb{P},\mathcal{B}(V)} \in L^q(\mathbb{P};V)\big) \in C(U, L^q(\mathbb{P};V))$ ensure that for all $q \in (2,\infty)$, $x \in U$ it holds that

$$\limsup_{U\ni u\to 0} \int_0^1 \mathbb{E}\big[\|\varphi'(X^{0,x} + \rho[X^{0,x+u} - X^{0,x}])\|_{L(V,W)}^q\big]\, d\rho \leq |c_{1,p}|^q \left(1 + 2\,\mathbb{E}\big[\|X^{0,x}\|_V^{pq}\big]\right) < \infty. \tag{22}$$

In addition, observe that the fact that $\forall q \in [1,\infty)\colon \big(U \ni x \mapsto [X^{0,x}]_{\mathbb{P},\mathcal{B}(V)} \in L^q(\mathbb{P};V)\big) \in C(U, L^q(\mathbb{P};V))$ shows that for all $x \in U$ it holds that

$$\limsup\nolimits_{U\ni y\to x} \mathbb{E}\big[\min\{1, \|X^{0,x} - X^{0,y}\|_V\}\big] = 0. \tag{23}$$

This implies that for all $\rho \in [0,1]$, $x \in U$ it holds that

$$\limsup\nolimits_{U\ni y\to x} \mathbb{E}\big[\min\{1, \|(X^{0,x} + \rho[X^{0,y} - X^{0,x}]) - X^{0,x}\|_V\}\big] = 0. \tag{24}$$

The fact that $\varphi' \in C(V, L(V,W))$ hence ensures that for all $\rho \in [0,1]$, $x \in U$ it holds that

$$\limsup\nolimits_{U\ni y\to x} \mathbb{E}\big[\min\{1, \|\varphi'(X^{0,x} + \rho[X^{0,y} - X^{0,x}]) - \varphi'(X^{0,x})\|_{L(V,W)}\}\big] = 0. \tag{25}$$

This and Lebesgue's theorem of dominated convergence imply that for all $x \in U$ it holds that

$$\limsup_{U\ni u\to 0} \int_0^1 \mathbb{E}\big[\min\{1, \|\varphi'(X^{0,x} + \rho[X^{0,x+u} - X^{0,x}]) - \varphi'(X^{0,x})\|_{L(V,W)}\}\big]\, d\rho = 0. \tag{26}$$

Combining this and, e.g., Lemma 4.2 in Hutzenthaler et al. [16] (with $I = \{\emptyset\}$, $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1]\times\Omega, \mathcal{B}([0,1]) \otimes \mathcal{F}, \mu_{[0,1]} \otimes \mathbb{P})$, $c = 1$, $X^n(\emptyset, (\rho,\omega)) = \|\varphi'(X^{0,x}(\omega) + \rho[X^{0,x+u_n}(\omega) - X^{0,x}(\omega)]) - \varphi'(X^{0,x}(\omega))\|_{L(V,W)}$ for $(\rho,\omega) \in [0,1]\times\Omega$, $n \in \mathbb{N}$, $x \in U$, $(u_m)_{m\in\mathbb{N}} \in \{v \in \mathbb{M}(\mathbb{N}, U)\colon \limsup_{m\to\infty} \|v_m\|_U = 0\}$ in the notation of Lemma 4.2 in Hutzenthaler et al. [16]) establishes that for all $\varepsilon \in (0,\infty)$, $x \in U$ and all sequences $(u_n)_{n\in\mathbb{N}} \subseteq U$ with $\limsup_{n\to\infty} \|u_n\|_U = 0$ it holds that

$$\limsup\nolimits_{n\to\infty} (\mu_{[0,1]}\otimes\mathbb{P})\big(\big\{(\rho,\omega) \in [0,1]\times\Omega\colon \|\varphi'(X^{0,x}(\omega) + \rho[X^{0,x+u_n}(\omega) - X^{0,x}(\omega)]) - \varphi'(X^{0,x}(\omega))\|_{L(V,W)} \geq \varepsilon\big\}\big) = 0. \tag{27}$$

This, (22), and, e.g., Proposition 4.5 in Hutzenthaler et al. [16] (with $I = \{\emptyset\}$, $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1]\times\Omega, \mathcal{B}([0,1]) \otimes \mathcal{F}, \mu_{[0,1]} \otimes \mathbb{P})$, $p = q$, $V = \mathbb{R}$, $X^n(\emptyset, (\rho,\omega)) = \|\varphi'(X^{0,x}(\omega) + \rho[X^{0,x+u_n}(\omega) - X^{0,x}(\omega)]) - \varphi'(X^{0,x}(\omega))\|_{L(V,W)}$ for $(\rho,\omega) \in [0,1]\times\Omega$, $n \in \mathbb{N}_0$, $x \in U$, $q \in (2,\infty)$, $(u_m)_{m\in\mathbb{N}_0} \in \{v \in \mathbb{M}(\mathbb{N}_0, U)\colon \limsup_{m\to\infty} \|v_m\|_U = \|v_0\|_U = 0\}$ in the notation of Proposition 4.5 in Hutzenthaler et al. [16]) yield that for all $x \in U$ and all sequences $(u_n)_{n\in\mathbb{N}_0} \subseteq U$ with $\limsup_{n\to\infty} \|u_n\|_U = \|u_0\|_U = 0$ it holds that

$$\limsup_{n\to\infty} \int_0^1 \mathbb{E}\big[\|\varphi'(X^{0,x} + \rho[X^{0,x+u_n} - X^{0,x}]) - \varphi'(X^{0,x})\|_{L(V,W)}^2\big]\, d\rho = 0. \tag{28}$$

Moreover, observe that the triangle inequality and the fact that $\forall\, q \in [1,\infty),\ x \in U\colon \big(U \ni u \mapsto [X^{1,(x,u)}]_{\mathbb{P},\mathcal{B}(V)} \in L^q(\mathbb{P};V)\big) \in L(U, L^q(\mathbb{P};V))$ assure that for all $x \in U$ it holds that

$$\begin{aligned}
&\limsup_{U\backslash\{0\}\ni u\to 0}\left[\frac{\|X^{0,x+u}-X^{0,x}\|_{\mathcal{L}^2(\mathbb{P};V)}}{\|u\|_U}\right]\\
&\le \limsup_{U\backslash\{0\}\ni u\to 0}\left[\frac{\|X^{0,x+u}-X^{0,x}-X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)}}{\|u\|_U}\right]+\sup_{u\in U\backslash\{0\}}\left[\frac{\|X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)}}{\|u\|_U}\right]\\
&= \sup_{u\in U\backslash\{0\}}\left[\frac{\|X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)}}{\|u\|_U}\right]<\infty.
\end{aligned} \tag{29}$$

Putting (28)–(29) into (20) yields that for all $x \in U$ it holds that

$$\begin{aligned}
&\limsup_{U\backslash\{0\}\ni u\to 0}\left(\frac{\|\mathbb{E}[\varphi(X^{0,x+u})-\varphi(X^{0,x})-\varphi'(X^{0,x})(X^{0,x+u}-X^{0,x})]\|_W}{\|u\|_U}\right)\\
&\le \limsup_{U\backslash\{0\}\ni u\to 0}\left(\frac{\|X^{0,x+u}-X^{0,x}\|_{\mathcal{L}^2(\mathbb{P};V)}}{\|u\|_U}\right)\\
&\cdot\left[\limsup_{U\backslash\{0\}\ni u\to 0}\int_0^1 \mathbb{E}\big[\|\varphi'(X^{0,x}+\rho[X^{0,x+u}-X^{0,x}])-\varphi'(X^{0,x})\|^2_{L(V,W)}\big]\,d\rho\right]^{1/2}=0.
\end{aligned} \tag{30}$$

Combining (17), (18), and (30) proves (16). In the next step we demonstrate that

$$\big(U \ni x \mapsto \big(U \ni u \mapsto \mathbb{E}[\varphi'(X^{0,x})X^{1,(x,u)}] \in W\big) \in L(U,W)\big) \in C(U, L(U,W)). \tag{31}$$

Observe that (21) and the fact that $\forall\, q \in [1,\infty)\colon \limsup_{U\ni y\to x}\mathbb{E}[\|X^{0,y}\|_V^q] = \mathbb{E}[\|X^{0,x}\|_V^q] < \infty$ ensure that for all $q \in (2,\infty)$, $\rho \in [0,1]$, $x \in U$ it holds that

$$\begin{aligned}
&\limsup\nolimits_{U\ni y\to x}\mathbb{E}\big[\|\varphi'(X^{0,x}+\rho[X^{0,y}-X^{0,x}])\|^q_{L(V,W)}\big]\\
&\le |c_{1,p}|^q\left(1+\mathbb{E}\big[\|X^{0,x}\|_V^{pq}\big]+\limsup\nolimits_{U\ni y\to x}\mathbb{E}\big[\|X^{0,y}\|_V^{pq}\big]\right)\\
&= |c_{1,p}|^q\left(1+2\,\mathbb{E}\big[\|X^{0,x}\|_V^{pq}\big]\right)<\infty.
\end{aligned} \tag{32}$$

Hence, we obtain that for all $q \in (2,\infty)$, $x \in U$ it holds that

$$\begin{aligned}
&\limsup_{U\ni y\to x}\mathbb{E}\big[\|\varphi'(X^{0,x})-\varphi'(X^{0,y})\|^q_{L(V,W)}\big]\\
&\le 2^q \limsup_{U\ni y\to x}\max_{\rho\in\{0,1\}}\mathbb{E}\big[\|\varphi'(X^{0,x}+\rho[X^{0,y}-X^{0,x}])\|^q_{L(V,W)}\big]<\infty.
\end{aligned} \tag{33}$$

Moreover, note that (25) (with $\rho = 1$ in the notation of (25)) and, e.g., Lemma 4.2 in Hutzenthaler et al. [16] (with $I = \{\emptyset\}$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $c = 1$, $X^n(\emptyset,\omega) = \|\varphi'(X^{0,u_n}(\omega)) - \varphi'(X^{0,u_0}(\omega))\|_{L(V,W)}$ for $\omega \in \Omega$, $n \in \mathbb{N}$, $(u_m)_{m\in\mathbb{N}_0} \in \{v \in \mathbb{M}(\mathbb{N}_0,U)\colon \limsup_{m\to\infty}\|v_m - v_0\|_U = 0\}$ in the notation of Lemma 4.2 in Hutzenthaler et al. [16]) establishes that for all $\varepsilon \in (0,\infty)$ and all sequences $(u_n)_{n\in\mathbb{N}_0} \subseteq U$ with $\limsup_{n\to\infty}\|u_n - u_0\|_U = 0$ it holds that

$$\limsup\nolimits_{n\to\infty}\mathbb{P}\big(\|\varphi'(X^{0,u_n})-\varphi'(X^{0,u_0})\|_{L(V,W)}\ge\varepsilon\big)=0. \tag{34}$$

Combining this, (33), and, e.g., Proposition 4.5 in Hutzenthaler et al. [16] (with $I = \{\emptyset\}$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $p = q$, $V = \mathbb{R}$, $X^n(\emptyset,\omega) = \|\varphi'(X^{0,u_n}(\omega)) - \varphi'(X^{0,u_0}(\omega))\|_{L(V,W)}$ for $\omega \in \Omega$, $q \in (2,\infty)$, $n \in \mathbb{N}_0$, $(u_m)_{m\in\mathbb{N}_0} \in \{v \in$

$\mathbb{M}(\mathbb{N}_0, U)\colon \limsup_{m\to\infty} \|v_m - v_0\|_U = 0\}$ in the notation of Proposition 4.5 in Hutzenthaler et al. [16]) yields that for all sequences $(u_n)_{n\in\mathbb{N}_0} \subseteq U$ with $\limsup_{n\to\infty} \|u_n - u_0\|_U = 0$ it holds that

$$\limsup_{n\to\infty} \mathbb{E}\big[\|\varphi'(X^{0,u_n}) - \varphi'(X^{0,u_0})\|^2_{L(V,W)}\big] = 0. \tag{35}$$

Next observe that the fact that for every $q \in [1,\infty)$ it holds that the function $U \ni x \mapsto \big(U \ni u \mapsto [X^{1,(x,u)}]_{\mathbb{P},\mathcal{B}(V)} \in L^q(\mathbb{P}; V)\big) \in L(U, L^q(\mathbb{P}; V))$ is continuous shows that for all $x \in U$ it holds that

$$\limsup_{U\ni y\to x} \sup_{u\in U,\, \|u\|_U=1} \|X^{1,(y,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} = \sup_{u\in U,\, \|u\|_U=1} \|X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} < \infty \tag{36}$$

and

$$\limsup_{U\ni y\to x} \sup_{u\in U,\, \|u\|_U=1} \|X^{1,(x,u)} - X^{1,(y,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} = 0. \tag{37}$$

Hölder's inequality and (33) hence ensure that for all $x \in U$ it holds that

$$\begin{aligned}
&\limsup_{U\ni y\to x} \sup_{u\in U,\, \|u\|_U=1} \big\|\mathbb{E}[\varphi'(X^{0,x})X^{1,(x,u)}] - \mathbb{E}[\varphi'(X^{0,y})X^{1,(y,u)}]\big\|_W \\
&\leq \limsup_{U\ni y\to x} \sup_{u\in U,\, \|u\|_U=1} \mathbb{E}\big[\|\varphi'(X^{0,x})(X^{1,(x,u)} - X^{1,(y,u)})\|_W\big] \\
&+ \limsup_{U\ni y\to x} \sup_{u\in U,\, \|u\|_U=1} \mathbb{E}\big[\|[\varphi'(X^{0,x}) - \varphi'(X^{0,y})]X^{1,(y,u)}\|_W\big] \\
&\leq \|\varphi'(X^{0,x})\|_{\mathcal{L}^2(\mathbb{P};L(V,W))} \limsup_{U\ni y\to x} \sup_{u\in U,\, \|u\|_U=1} \|X^{1,(x,u)} - X^{1,(y,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} \\
&+ \left[\limsup_{U\ni y\to x} \|\varphi'(X^{0,x}) - \varphi'(X^{0,y})\|_{\mathcal{L}^2(\mathbb{P};L(V,W))}\right] \limsup_{U\ni y\to x} \sup_{u\in U,\, \|u\|_U=1} \|X^{1,(y,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} \\
&\leq c_{1,p}\, \|\max\{1, \|X^{0,x}\|_V\}\|^p_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \limsup_{U\ni y\to x} \sup_{u\in U,\, \|u\|_U=1} \|X^{1,(x,u)} - X^{1,(y,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} \\
&+ \left[\limsup_{U\ni y\to x} \|\varphi'(X^{0,x}) - \varphi'(X^{0,y})\|_{\mathcal{L}^2(\mathbb{P};L(V,W))}\right] \sup_{u\in U,\, \|u\|_U=1} \|X^{1,(x,u)}\|_{\mathcal{L}^2(\mathbb{P};V)} = 0.
\end{aligned} \tag{38}$$

This proves (31). Combining (15), (16), and (31) establishes item (ii) and item (iii). The proof of Lemma 2.1 is thus completed. □

Lemma 2.2 (Pointwise differentiation) *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be $\mathbb{R}$-Banach spaces with $\#_V > 1$ and let $n \in \mathbb{N}$, $f \in C^n(V, W)$, $g \in C(V, L^{(n+1)}(V, W))$ satisfy for all $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in V^n$, $x \in V$ that*

$$\limsup_{V\setminus\{0\}\ni h\to 0} \left[\frac{\|f^{(n)}(x+h)\mathbf{u} - f^{(n)}(x)\mathbf{u} - g(x)(u_1, u_2, \ldots, u_n, h)\|_W}{\|h\|_V}\right] = 0. \tag{39}$$

Then it holds that $f \in C^{n+1}(V, W)$ and $f^{(n+1)} = g$.

Proof We first note that (39) and the fact that $\forall\, x, u_1, u_2, \ldots, u_n \in V\colon \big(V \ni h \mapsto g(x)(u_1, u_2, \ldots, u_n, h) \in W\big) \in L(V, W)$ and $\big(V \ni y \mapsto \big(V \ni h \mapsto g(y)(u_1, u_2, \ldots, u_n, h) \in W\big) \in L(V, W)\big) \in C(V, L(V, W))$ imply that for all $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in V^n$, $x, h \in V$ it holds that $\big(V \ni y \mapsto f^{(n)}(y)\mathbf{u} \in W\big) \in C^1(V, W)$ and

$\left(\frac{d}{dx}\left(f^{(n)}(x)\mathbf{u}\right)\right)h = g(x)(u_1, u_2, \ldots, u_n, h)$. This and the fundamental theorem of calculus imply that for all $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in V^n$, $x, h \in V$ it holds that

$$\begin{aligned}
&\|f^{(n)}(x+h)\mathbf{u} - f^{(n)}(x)\mathbf{u} - g(x)(u_1, u_2, \ldots, u_n, h)\|_W \\
&= \left\|\int_0^1 \left[g(x+\rho h) - g(x)\right](u_1, u_2, \ldots, u_n, h)\, d\rho\right\|_W \\
&\leq \|h\|_V \left[\prod_{i=1}^n \|u_i\|_V\right] \int_0^1 \|g(x+\rho h) - g(x)\|_{L^{(n+1)}(V,W)}\, d\rho.
\end{aligned} \tag{40}$$

In addition, observe that the assumption that $g \in C(V, L^{(n+1)}(V, W))$ ensures that for all $x \in V$ it holds that

$$\limsup_{V \ni h \to 0} \sup_{\rho \in [0,1]} \|g(x+\rho h)\|_{L^{(n+1)}(V,W)} < \infty. \tag{41}$$

Lebesgue's theorem of dominated convergence therefore ensures that for all $x \in V$ it holds that

$$\limsup_{V \ni h \to 0} \int_0^1 \|g(x+\rho h) - g(x)\|_{L^{(n+1)}(V,W)}\, d\rho = 0. \tag{42}$$

Combining (40) with (42) yields that for all $x \in V$ it holds that

$$\limsup_{V\setminus\{0\} \ni h \to 0} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_n)\in(V\setminus\{0\})^n} \left[\frac{\|f^{(n)}(x+h)\mathbf{u} - f^{(n)}(x)\mathbf{u} - g(x)(u_1, u_2, \ldots, u_n, h)\|_W}{\|h\|_V \prod_{i=1}^n \|u_i\|_V}\right] = 0. \tag{43}$$

This and the assumption that $g \in C(V, L^{(n+1)}(V, W))$ complete the proof of Lemma 2.2. □

3 Regularity of Transition Semigroups for Stochastic Evolution Equations

This section establishes regularity properties of the transition semigroup.

Lemma 3.1 *Let* $(V, \|\cdot\|_V)$ *and* $(W, \|\cdot\|_W)$ *be* $\mathbb{R}$*-Banach spaces with* $\#_V > 1$*, let* $n \in \mathbb{N}$*,* $\varphi \in C^{n+1}(V, W)$*, and let* $\Phi\colon V^{n+1} \to W$ *be the function which satisfies for all* $\mathbf{v} = (v_1, v_2, \ldots, v_{n+1}) \in V^{n+1}$ *that* $\Phi(\mathbf{v}) = \varphi^{(n)}(v_{n+1})(v_1, v_2, \ldots, v_n)$*. Then it holds for all* $\mathbf{v} = (v_1, v_2, \ldots, v_{n+1})$*,* $\mathbf{h} = (h_1, h_2, \ldots, h_{n+1}) \in V^{n+1}$ *that* $\Phi \in C^1(V^{n+1}, W)$ *and*

$$\begin{aligned}
\Phi'(\mathbf{v})\mathbf{h} = &\varphi^{(n+1)}(v_{n+1})(v_1, v_2, \ldots, v_n, h_{n+1}) \\
&+ \sum_{i=1}^n \varphi^{(n)}(v_{n+1})(v_1, v_2, \ldots, v_{i-1}, h_i, v_{i+1}, v_{i+2}, \ldots, v_n).
\end{aligned} \tag{44}$$

Proof Throughout this proof let $P\colon V^{n+1} \to L^{(n)}(V, W) \times V^n$, $\beta\colon L^{(n)}(V, W) \times V^n \to W$, and $\phi\colon V^2 \to L^{(n)}(V, W)$ be the functions which satisfy for all $A \in L^{(n)}(V, W)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V^n$, $v, h \in V$ that

$$P(v_1, v_2, \ldots, v_n, v) = (\varphi^{(n)}(v), \mathbf{v}), \qquad \beta(A, \mathbf{v}) = A(v_1, v_2, \ldots, v_n), \tag{45}$$

and

$$\phi(v, h)\mathbf{v} = \varphi^{(n+1)}(v)(v_1, v_2, \ldots, v_n, h). \tag{46}$$

We note that for all $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V^n$, $v \in V$ it holds that

$$\Phi(v_1, v_2, \ldots, v_n, v) = \beta(P(v_1, v_2, \ldots, v_n, v)). \tag{47}$$

Furthermore, observe that the assumption that $\varphi \in C^{n+1}(V, W)$ ensures that for all $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $\mathbf{h} = (h_1, h_2, \ldots, h_n) \in V^n$, $v, h \in V$ it holds that $P \in C^1(V^{n+1}, L^{(n)}(V, W) \times V^n)$ and

$$P'(v_1, v_2, \ldots, v_n, v)(h_1, h_2, \ldots, h_n, h) = (\phi(v, h), \mathbf{h}). \tag{48}$$

Moreover, the fact that β is an $(n+1)$-multilinear and continuous function and, e.g., Theorem 3.7 in Coleman [7] assure that for all $A, \tilde{A} \in L^{(n)}(V, W)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $\mathbf{h} = (h_1, h_2, \ldots, h_n) \in V^n$ it holds that $\beta \in C^1(L^{(n)}(V, W) \times V^n, W)$ and

$$\beta'(A, \mathbf{v})(\tilde{A}, \mathbf{h}) = \tilde{A}(v_1, v_2, \ldots, v_n) + \sum_{i=1}^{n} A(v_1, v_2, \ldots, v_{i-1}, h_i, v_{i+1}, v_{i+2}, \ldots, v_n). \tag{49}$$

Combining (47)–(49) with the chain rule yields that for all $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $\mathbf{h} = (h_1, h_2, \ldots, h_n) \in V^n$, $v, h \in V$ it holds that $\Phi \in C^1(V^{n+1}, W)$ and

$$\begin{aligned}
&\Phi'(v_1, v_2, \ldots, v_n, v)(h_1, h_2, \ldots, h_n, h) \\
&= \beta'(P(v_1, v_2, \ldots, v_n, v))P'(v_1, v_2, \ldots, v_n, v)(h_1, h_2, \ldots, h_n, h) \\
&= \beta'(P(v_1, v_2, \ldots, v_n, v))(\phi(v, h), \mathbf{h}) \\
&= \beta'(\varphi^{(n)}(v), \mathbf{v})(\phi(v, h), \mathbf{h}) \\
&= \phi(v, h)\mathbf{v} + \sum_{i=1}^{n} \varphi^{(n)}(v)(v_1, v_2, \ldots, v_{i-1}, h_i, v_{i+1}, v_{i+2}, \ldots, v_n) \\
&= \varphi^{(n+1)}(v)(v_1, v_2, \ldots, v_n, h) + \sum_{i=1}^{n} \varphi^{(n)}(v)(v_1, v_2, \ldots, v_{i-1}, h_i, v_{i+1}, v_{i+2}, \ldots, v_n).
\end{aligned} \tag{50}$$

This implies (44). The proof of Lemma 3.1 is thus completed. □

Lemma 3.2 *Assume the setting in* Section 1.2*, let $n \in \mathbb{N}$, $\varphi \in C_b^n(H, V)$, $F \in C_b^n(H, H)$, $B \in C_b^n(H, HS(U, H))$, let $X^{k,\mathbf{u}} \colon [0, T] \times \Omega \to H$, $\mathbf{u} \in H^{k+1}$, $k \in \{0, 1, \ldots, n\}$, be $(\mathcal{F}_t)_{t \in [0,T]}$/$\mathcal{B}(H)$-predictable stochastic processes which satisfy for all $k \in \{0, 1, \ldots, n\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_k) \in H^{k+1}$, $p \in (0, \infty)$, $t \in [0, T]$ that* $\sup_{s \in [0,T]} \mathbb{E}\big[\|X_s^{k,\mathbf{u}}\|_H^p\big] < \infty$ *and*

$$\begin{aligned}
&[X_t^{k,\mathbf{u}} - e^{tA}\, \mathbb{1}_{\{0,1\}}(k)\, u_k]_{\mathbb{P},\mathcal{B}(H)} \\
&= \int_0^t e^{(t-s)A} \bigg[\mathbb{1}_{\{0\}}(k)\, F(X_s^{0,u_0}) \\
&\quad + \sum_{\varpi \in \Pi_k} F^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi}, [\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi}, [\mathbf{u}]_2^\varpi}, \ldots, X_s^{\#_{I_{\#_\varpi}^\varpi}, [\mathbf{u}]_{\#_\varpi}^\varpi}\big) \bigg] ds \\
&\quad + \int_0^t e^{(t-s)A} \bigg[\mathbb{1}_{\{0\}}(k)\, B(X_s^{0,u_0}) \\
&\quad + \sum_{\varpi \in \Pi_k} B^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi}, [\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi}, [\mathbf{u}]_2^\varpi}, \ldots, X_s^{\#_{I_{\#_\varpi}^\varpi}, [\mathbf{u}]_{\#_\varpi}^\varpi}\big) \bigg] dW_s,
\end{aligned} \tag{51}$$

and let $\phi \colon [0, T] \times H \to V$ be the function which satisfies for all $t \in [0, T]$, $x \in H$ that $\phi(t, x) = \mathbb{E}[\varphi(X_t^{0,x})]$. Then

(i) *it holds for all* $k \in \{1, 2, \ldots, n\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_k) \in H^{k+1}$, $t \in [0, T]$ *that*

$$\sum_{\varpi \in \Pi_k} \mathbb{E}\Big[\big\|\varphi^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\big\|_V\Big] < \infty, \quad (52)$$

(ii) *it holds for all* $t \in [0, T]$ *that* $\big(H \ni x \mapsto \phi(t, x) \in V\big) \in C_b^n(H, V)$,

(iii) *it holds for all* $k \in \{1, 2, \ldots, n\}$, $\mathbf{u} \in H^k$, $x \in H$, $t \in [0, T]$ *that*

$$\begin{aligned}&\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t, x)\mathbf{u}\\ &= \sum_{\varpi \in \Pi_k} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big],\end{aligned} \quad (53)$$

(iv) *it holds for all* $p \in (0, \infty)$, $k \in \{1, 2, \ldots, n\}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in [0, 1/2)^k$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *that*

$$\sup_{x \in H} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \sup_{t \in (0,T]} \left[\frac{t^{\iota_{\mathbb{N}}^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{k,(x,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right] < \infty, \quad (54)$$

(v) *it holds for all* $k \in \{1, 2, \ldots, n\}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in [0, 1/2)^k$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *that*

$$\begin{aligned}&\sup_{v \in H} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \sup_{t \in (0,T]} \left[\frac{t^{\sum_{i=1}^k \delta_i} \big\|\big(\frac{\partial^k}{\partial x^k}\phi\big)(t, v)\mathbf{u}\big\|_V}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right]\\ &\leq |T \vee 1|^{\lfloor k/2 \rfloor \min\{1-\alpha, 1/2-\beta\}} \|\varphi\|_{C_b^k(H,V)}\\ &\cdot \sum_{\varpi \in \Pi_k} \prod_{I \in \varpi} \sup_{x \in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t \in (0,T]} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i \in I} \|u_i\|_{H_{-\delta_i}}}\right] < \infty,\end{aligned} \quad (55)$$

(vi) *it holds for all* $p \in (0, \infty)$ *that*

$$\sup_{\substack{x,y\in H,\\ x \neq y}} \sup_{t \in (0,T]} \left[\frac{\|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x - y\|_H}\right] < \infty, \quad (56)$$

(vii) *it holds for all* $p \in (0, \infty)$, $k \in \{1, 2, \ldots, n\}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in [0, 1/2)^k$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *and* $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} < \infty$ *that*

$$\sup_{\substack{x,y\in H,\\ x \neq y}} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \sup_{t \in (0,T]} \left[\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x - y\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right] < \infty, \quad (57)$$

and

(viii) *it holds for all* $k \in \{1, 2, \ldots, n\}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in [0, 1/2)^k$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$ *with* $\sum_{i=1}^{k} \delta_i < 1/2$ *and* $|F|_{\mathrm{Lip}^k(H, H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H, HS(U, H_{-\beta}))} + |\varphi|_{\mathrm{Lip}^k(H,V)} < \infty$ *that*

$$
\begin{aligned}
&\sup_{\substack{v,w\in H,\\ v\neq w}} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \left[\frac{t^{\sum_{i=1}^{k}\delta_i} \big\| \big[\big(\frac{\partial^k}{\partial x^k}\phi\big)(t, v) - \big(\frac{\partial^k}{\partial x^k}\phi\big)(t, w) \big] \mathbf{u} \big\|_V}{\|v - w\|_H \prod_{i=1}^{k} \|u_i\|_{H_{-\delta_i}}} \right] \\
&\leq |T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}} \, \|\varphi\|_{\mathrm{Lip}^k(H,V)} \\
&\cdot \sum_{\varpi\in\Pi_k} \Bigg\{ \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{t\in(0,T]} \left[\frac{\|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^{\#_\varpi+1}(\mathbb{P};H)}}{\|x-y\|_H} \right] \\
&\cdot \prod_{I\in\varpi} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t\in(0,T]} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi+1}(\mathbb{P};H)}}{\prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \right] \\
&+ \sum_{I\in\varpi} \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t\in(0,T]} \left[\frac{t^{\iota_{I\cup\{k+1\}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{\#_I,(x,\mathbf{u})} - X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\|x-y\|_H \prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \right] \\
&\cdot \prod_{J\in\varpi\setminus\{I\}} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in J}\in(H\setminus\{0\})^{\#_J}} \sup_{t\in(0,T]} \left[\frac{t^{\iota_J^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\#_J,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in J} \|u_i\|_{H_{-\delta_i}}} \right] \Bigg\} < \infty.
\end{aligned} \tag{58}
$$

Proof Throughout this proof let $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$ and let $\mathbb{D}_k \in \mathcal{P}(\mathbb{R}^k)$, $k \in \mathbb{N}$, be the sets which satisfy for all $k \in \mathbb{N}$ that $\mathbb{D}_k = \{(\delta_1, \delta_2, \ldots, \delta_k) \in [0, 1/2)^k : \sum_{i=1}^{k} \delta_i < 1/2\}$. Note that Hölder's inequality shows that for all $k \in \{1, 2, \ldots, n\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_k) \in H^{k+1}$, $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\sum_{\varpi\in\Pi_k} \mathbb{E}\Big[\big\| \varphi^{(\#_\varpi)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\mathbf{u}]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\mathbf{u}]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\mathbf{u}]_{\#_\varpi}^\varpi} \big) \big\|_V \Big] \\
&\leq \sum_{\varpi\in\Pi_k} |\varphi|_{C_b^{\#_\varpi}(H,V)} \prod_{i=1}^{\#_\varpi} \|X_t^{\#_{I_i^\varpi}, [\mathbf{u}]_i^\varpi}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}.
\end{aligned} \tag{59}
$$

This, the assumption that $\varphi \in C_b^n(H, V)$, and the assumption that $\forall\, k \in \{1, 2, \ldots, n\}$, $\mathbf{u} \in H^{k+1}$, $p \in (0, \infty)$: $\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t^{k,\mathbf{u}}\|_H^p\big] < \infty$ establish item (i). Moreover, note that (59) implies that for all $k \in \{1, 2, \ldots, n\}$, $\delta_1, \delta_2, \ldots, \delta_k \in [0, \infty)$, $\mathbf{u} = (u_1, u_2, \ldots, u_k) \in (H \setminus \{0\})^k$, $x \in H$, $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\frac{1}{\prod_{i=1}^{k} \|u_i\|_{H_{-\delta_i}}} \sum_{\varpi\in\Pi_k} \mathbb{E}\Big[\big\| \varphi^{(\#_\varpi)}(X_t^{0,x}) \big(X_t^{\#_{I_1^\varpi}, [(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [(x,\mathbf{u})]_{\#_\varpi}^\varpi} \big) \big\|_V \Big] \\
&\leq \sum_{\varpi\in\Pi_k} |\varphi|_{C_b^{\#_\varpi}(H,V)} \prod_{i=1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi}, [(x,\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{j=1}^{\#_{I_i^\varpi}} \|u_{I_{i,j}^\varpi}\|_{H_{-\delta_{I_{i,j}^\varpi}}}}.
\end{aligned} \tag{60}
$$

In addition, (51) and item (ii) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = 0$, $\beta = 0$, $k = k$, $p = p$, $\boldsymbol{\delta} = (0, 0, \ldots, 0) \in \mathbb{R}^k$ for $p \in [2, \infty)$, $k \in \{1, 2, \ldots, n\}$ in the notation of Theorem 2.1 in [2]) ensure that for all $k \in \{1, 2, \ldots, n\}$, $p \in [2, \infty)$, $t \in [0, T]$ it holds that

$$\sup_{\mathbf{u}=(u_0,u_1,\ldots,u_k)\in H\times(H\setminus\{0\})^k} \left[\frac{\|X_t^{k,\mathbf{u}}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k \|u_i\|_H} \right] < \infty. \tag{61}$$

This, Jensen's inequality, and (60) (with $k = k$, $\delta_1 = 0$, $\delta_2 = 0, \ldots, \delta_k = 0$ for $k \in \{1, 2, \ldots, n\}$ in the notation of (60)) imply that for all $k \in \{1, 2, \ldots, n\}$, $t \in [0, T]$ it holds that

$$\begin{aligned} &\sup_{x\in H} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \Bigg(\frac{1}{\prod_{i=1}^k \|u_i\|_H} \\ &\cdot \sum_{\varpi\in\Pi_k} \mathbb{E}\Big[\big\| \varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big) \big\|_V \Big] \Bigg) \\ &\leq \sum_{\varpi\in\Pi_k} |\varphi|_{C_b^{\#_\varpi}(H,V)} \prod_{I\in\varpi} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left(\frac{\|X_t^{\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_H} \right) < \infty. \end{aligned} \tag{62}$$

Furthermore, (51) and item (iii) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = 0$, $\beta = 0$, $k = k$, $p = p$, $x = x$ for $x \in H$, $p \in [2, \infty)$, $k \in \{1, 2, \ldots, n\}$ in the notation of Theorem 2.1 in [2]) assure that for all $k \in \{1, 2, \ldots, n\}$, $p \in [2, \infty)$, $x \in H$, $t \in [0, T]$ it holds that

$$\big(H^k \ni \mathbf{u} \mapsto [X_t^{k,(x,\mathbf{u})}]_{\mathbb{P},\mathcal{B}(H)} \in L^p(\mathbb{P}; H)\big) \in L^{(k)}(H, L^p(\mathbb{P}; H)). \tag{63}$$

This and (62) establish that for all $k \in \{1, 2, \ldots, n\}$, $x \in H$, $t \in [0, T]$ it holds that

$$\begin{aligned} &\Big(H^k \ni \mathbf{u} \mapsto \textstyle\sum_{\varpi\in\Pi_k} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big] \in V\Big) \\ &\qquad \in L^{(k)}(H, V). \end{aligned} \tag{64}$$

In the next step we prove that for all $k \in \{1, 2, \ldots, n\}$, $t \in [0, T]$ it holds that

$$\begin{aligned} &\Big(H \ni x \mapsto \Big(H^k \ni \mathbf{u} \mapsto \textstyle\sum_{\varpi\in\Pi_k} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, \\ &\qquad X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big] \in V\Big) \in L^{(k)}(H, V)\Big) \in C(H, L^{(k)}(H, V)). \end{aligned} \tag{65}$$

For this we observe that the triangle inequality and Hölder's inequality imply that for all $k \in \{1, 2, \ldots, n\}$, $\delta_1, \delta_2, \ldots, \delta_k \in [0, \infty)$, $\mathbf{u} = (u_1, u_2, \ldots, u_k) \in (H \setminus \{0\})^k$, $x, y \in H$, $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\frac{1}{\prod_{i=1}^{k} \|u_i\|_{H_{-\delta_i}}} \Big\| \sum_{\varpi \in \Pi_k} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big) \\
&\quad - \varphi^{(\#_\varpi)}(X_t^{0,y})\big(X_t^{\#_{I_1^\varpi},[(y,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(y,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(y,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big]\Big\|_V \\
&\leq \frac{1}{\prod_{i=1}^{k} \|u_i\|_{H_{-\delta_i}}} \sum_{\varpi \in \Pi_k} \Big(\mathbb{E}\Big[\big\|\big[\varphi^{(\#_\varpi)}(X_t^{0,x}) - \varphi^{(\#_\varpi)}(X_t^{0,y})\big]\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, \\
&\quad X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\big\|_V\Big] + \mathbb{E}\Big[\big\|\varphi^{(\#_\varpi)}(X_t^{0,y})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big) \\
&\quad - \varphi^{(\#_\varpi)}(X_t^{0,y})\big(X_t^{\#_{I_1^\varpi},[(y,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(y,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(y,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\big\|_V\Big]\Big) \\
&\leq \sum_{\varpi \in \Pi_k} \Bigg\{ \|\varphi^{(\#_\varpi)}(X_t^{0,x}) - \varphi^{(\#_\varpi)}(X_t^{0,y})\|_{\mathcal{L}^{\#_\varpi+1}(\mathbb{P}; L^{(\#_\varpi)}(H,V))} \prod_{i=1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi},[(x,\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{\#_\varpi+1}(\mathbb{P}; H)}}{\prod_{j=1}^{\#_{I_i^\varpi}} \|u_{I_{i,j}^\varpi}\|_{H_{-\delta_{I_{i,j}^\varpi}}}} \\
&\quad + |\varphi|_{C_b^{\#_\varpi}(H,V)} \sum_{i=1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi},[(x,\mathbf{u})]_i^\varpi} - X_t^{\#_{I_i^\varpi},[(y,\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P}; H)}}{\prod_{j=1}^{\#_{I_i^\varpi}} \|u_{I_{i,j}^\varpi}\|_{H_{-\delta_{I_{i,j}^\varpi}}}} \\
&\quad \cdot \Bigg[\prod_{j=1}^{i-1} \frac{\|X_t^{\#_{I_j^\varpi},[(y,\mathbf{u})]_j^\varpi}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P}; H)}}{\prod_{l=1}^{\#_{I_j^\varpi}} \|u_{I_{j,l}^\varpi}\|_{H_{-\delta_{I_{j,l}^\varpi}}}}\Bigg]\Bigg[\prod_{j=i+1}^{\#_\varpi} \frac{\|X_t^{\#_{I_j^\varpi},[(x,\mathbf{u})]_j^\varpi}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P}; H)}}{\prod_{l=1}^{\#_{I_j^\varpi}} \|u_{I_{j,l}^\varpi}\|_{H_{-\delta_{I_{j,l}^\varpi}}}}\Bigg]\Bigg\}.
\end{aligned}
\tag{66}
$$

Next we note that (51) and item (iii) of Corollary 2.10 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $\alpha = 0$, $\beta = 0$, $W = W$, $A = A$, $F = F$, $B = B$, $p = p$, $\delta = 0$ for $p \in [2, \infty)$ in the notation of Corollary 2.10 in [1]) ensure that for all $p \in [2, \infty)$ it holds that

$$
\sup_{x,y \in H,\, x \neq y} \sup_{t \in [0,T]} \frac{\|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^p(\mathbb{P}; H)}}{\|x - y\|_H} < \infty. \tag{67}
$$

This implies that for all $x \in H$, $t \in [0, T]$ it holds that

$$
\limsup_{H \ni y \to x} \mathbb{E}[\min\{1, \|X_t^{0,x} - X_t^{0,y}\|_H\}] = 0. \tag{68}
$$

The fact that $\forall\, k \in \{1, 2, \ldots, n\}$: $\varphi^{(k)} \in C(H, L^{(k)}(H, V))$ therefore assures that for all $k \in \{1, 2, \ldots, n\}$, $x \in H$, $t \in [0, T]$ it holds that

$$
\limsup_{H \ni y \to x} \mathbb{E}[\min\{1, \|\varphi^{(k)}(X_t^{0,x}) - \varphi^{(k)}(X_t^{0,y})\|_{L^{(k)}(H,V)}\}] = 0. \tag{69}
$$

Combining this and, e.g., Lemma 4.2 in Hutzenthaler et al. [16] (with $I = \{\emptyset\}$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $c = 1$, $X^m(\emptyset, \omega) = \|\varphi^{(k)}(X_t^{0,y_m}(\omega)) - \varphi^{(k)}(X_t^{0,y_0}(\omega))\|_{L^{(k)}(H,V)}$ for $\omega \in \Omega$,

$t \in [0,T]$, $m \in \mathbb{N}$, $k \in \{1,2,\dots,n\}$, $(y_l)_{l\in\mathbb{N}_0} \in \{v \in \mathbb{M}(\mathbb{N}_0, H)\colon \limsup_{l\to\infty} \|v_l - v_0\|_H = 0\}$ in the notation of Lemma 4.2 in Hutzenthaler et al. [16]) establishes that for all $k \in \{1,2,\dots,n\}$, $\varepsilon \in (0,\infty)$, $t \in [0,T]$ and all sequences $(y_m)_{m\in\mathbb{N}_0} \subseteq H$ with $\limsup_{m\to\infty} \|y_m - y_0\|_H = 0$ it holds that

$$\limsup_{m\to\infty} \mathbb{P}\big(\|\varphi^{(k)}(X_t^{0,y_m}) - \varphi^{(k)}(X_t^{0,y_0})\|_{L^{(k)}(H,V)} \ge \varepsilon\big) = 0. \tag{70}$$

Combining this, the fact that $\forall k \in \{1,2,\dots,n\}\colon \sup_{x\in H} \|\varphi^{(k)}(x)\|_{L^{(k)}(H,V)} < \infty$, and, e.g., Proposition 4.5 in Hutzenthaler et al. [16] (with $I = \{\emptyset\}$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $p = p$, $V = \mathbb{R}$, $X^m(\emptyset, \omega) = \|\varphi^{(k)}(X_t^{0,y_m}(\omega)) - \varphi^{(k)}(X_t^{0,y_0}(\omega))\|_{L^{(k)}(H,V)}$ for $\omega \in \Omega$, $t \in [0,T]$, $p \in [2,\infty)$, $k \in \{1,2,\dots,n\}$, $m \in \mathbb{N}_0$, $(y_l)_{l\in\mathbb{N}_0} \in \{v \in \mathbb{M}(\mathbb{N}_0, H)\colon \limsup_{l\to\infty} \|v_l - v_0\|_H = 0\}$ in the notation of Proposition 4.5 in Hutzenthaler et al. [16]) therefore shows that for all $k \in \{1,2,\dots,n\}$, $p \in [2,\infty)$, $t \in [0,T]$ and all sequences $(y_m)_{m\in\mathbb{N}_0} \subseteq H$ with $\limsup_{m\to\infty} \|y_m - y_0\|_H = 0$ it holds that

$$\limsup_{m\to\infty} \mathbb{E}\Big[\|\varphi^{(k)}(X_t^{0,y_m}) - \varphi^{(k)}(X_t^{0,y_0})\|^p_{L^{(k)}(H,V)}\Big] = 0. \tag{71}$$

Furthermore, (51) and item (v) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = 0$, $\beta = 0$, $k = k$, $p = p$ for $p \in [2,\infty)$, $k \in \{1,2,\dots,n\}$ in the notation of Theorem 2.1 in [2]) ensure that for all $k \in \{1,2,\dots,n\}$, $p \in [2,\infty)$, $t \in [0,T]$ it holds that

$$\begin{aligned}\Big(H \ni x \mapsto \big(H^k \ni \mathbf{u} \mapsto [X_t^{k,(x,\mathbf{u})}]_{\mathbb{P},\mathcal{B}(H)} \in L^p(\mathbb{P}; H)\big) \in L^{(k)}(H, L^p(\mathbb{P}; H))\Big)&\\ \in C(H, L^{(k)}(H, L^p(\mathbb{P}; H))).&\end{aligned} \tag{72}$$

Combining (66) (with $k = k$, $\delta_1 = 0$, $\delta_2 = 0, \dots, \delta_k = 0$ for $k \in \{1,2,\dots,n\}$ in the notation of (66)) with (61), (71), (72), and Jensen's inequality yields that for all $k \in \{1,2,\dots,n\}$, $x \in H$, $t \in [0,T]$ it holds that

$$\begin{aligned}&\limsup_{H\ni y\to x} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \Bigg(\frac{1}{\prod_{i=1}^k \|u_i\|_H}\\
&\cdot \Bigg\| \sum_{\varpi\in\Pi_k} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I^\varpi_{\#_\varpi}},[(x,\mathbf{u})]^\varpi_{\#_\varpi}}\big)\\
&- \varphi^{(\#_\varpi)}(X_t^{0,y})\big(X_t^{\#_{I_1^\varpi},[(y,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(y,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I^\varpi_{\#_\varpi}},[(y,\mathbf{u})]^\varpi_{\#_\varpi}}\big)\Big]\Bigg\|_V\Bigg)\\
&\le \sum_{\varpi\in\Pi_k} \Bigg\{\Bigg(\limsup_{H\ni y\to x} \Big[\|\varphi^{(\#_\varpi)}(X_t^{0,x}) - \varphi^{(\#_\varpi)}(X_t^{0,y})\|_{\mathcal{L}^{\#_\varpi+1}(\mathbb{P}; L^{(\#_\varpi)}(H,V))}\Big]\Bigg)\\
&\cdot \Bigg(\prod_{I\in\varpi} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \Bigg[\frac{\|X_t^{\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi+1}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_H}\Bigg]\Bigg)\\
&+ |\varphi|_{C_b^{\#_\varpi}(H,V)} \sum_{I\in\varpi} \Bigg(\limsup_{H\ni y\to x} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \Bigg[\frac{\|X_t^{\#_I,(x,\mathbf{u})} - X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_H}\Bigg]\Bigg)\\
&\cdot \Bigg(\prod_{J\in\varpi\setminus\{I\}} \sup_{y\in H} \sup_{\mathbf{u}=(u_i)_{i\in J}\in(H\setminus\{0\})^{\#_J}} \Bigg[\frac{\|X_t^{\#_J,(y,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in J}\|u_i\|_H}\Bigg]\Bigg)\Bigg\} = 0.\end{aligned} \tag{73}$$

This proves (65). Next we claim that for all $k \in \{1, 2, \dots, n\}$, $\mathbf{u} \in H^k$, $x \in H$, $t \in [0, T]$ it holds that $\big(H \ni y \mapsto \phi(t, y) \in V\big) \in C_b^k(H, V)$ and

$$\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t, x)\mathbf{u} = \sum_{\varpi \in \Pi_k} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big]. \tag{74}$$

We now prove (74) by induction on $k \in \{1, 2, \dots, n\}$. For the base case $k = 1$ we note that (51), Jensen's inequality, and items (ix)–(x) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = 0$, $\beta = 0$, $p = p$, $t = t$ for $t \in [0, T]$, $p \in [2, \infty)$ in the notation of Theorem 2.1 in [2]) ensure that for all $p \in [1, \infty)$, $x, u_1 \in H$, $t \in [0, T]$ it holds that $\big(H \ni y \mapsto [X_t^{0,y}]_{\mathbb{P},\mathcal{B}(H)} \in L^p(\mathbb{P}; H)\big) \in C^1(H, L^p(\mathbb{P}; H))$ and

$$\big(\tfrac{d}{dx}[X_t^{0,x}]_{\mathbb{P},\mathcal{B}(H)}\big)u_1 = [X_t^{1,(x,u_1)}]_{\mathbb{P},\mathcal{B}(H)}. \tag{75}$$

Lemma 2.1 (with $U = H$, $V = H$, $W = V$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $X^{m,\mathbf{u}} = X_t^{m,\mathbf{u}}$, $\varphi = \varphi$ for $t \in [0, T]$, $\mathbf{u} \in H^{m+1}$, $m \in \{0, 1\}$ in the notation of Lemma 2.1) therefore implies that for all $x, u \in H$, $t \in [0, T]$ it holds that

$$\big(H \ni y \mapsto \phi(t, y) = \mathbb{E}[\varphi(X_t^{0,y})] \in V\big) \in C^1(H, V) \tag{76}$$

and

$$\big(\tfrac{\partial}{\partial x}\phi\big)(t, x)u = \mathbb{E}[\varphi'(X_t^{0,x})X_t^{1,(x,u)}]. \tag{77}$$

This and (62) prove (74) in the base case $k = 1$. For the induction step $\{1, 2, \dots, n-1\} \ni k \to k+1 \in \{2, 3, \dots, n\}$ assume that there exists a natural number $k \in \{1, 2, \dots, n-1\}$ such that (74) holds for $k = 1, k = 2, \dots, k = k$, let $\Phi_m \colon H^{m+1} \to V$, $m \in \{1, 2, \dots, k\}$, be the functions which satisfy for all $m \in \{1, 2, \dots, k\}$, $\mathbf{u} = (u_1, u_2, \dots, u_{m+1}) \in H^{m+1}$ that $\Phi_m(\mathbf{u}) = \varphi^{(m)}(u_{m+1})(u_1, u_2, \dots, u_m)$, and let $\mathbf{Y}^{m,\mathbf{v},\varpi,\mathbf{u}} \colon [0, T] \times \Omega \to H^{\#_\varpi+1}$, $\mathbf{u} \in H^k$, $\varpi \in \Pi_k$, $\mathbf{v} \in H^{m+1}$, $m \in \{0, 1\}$, be the stochastic processes which satisfy for all $\varpi \in \Pi_k$, $\mathbf{u} \in H^k$, $x, h \in H$, $t \in [0, T]$ that

$$\mathbf{Y}_t^{0,x,\varpi,\mathbf{u}} = \big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,x}\big) \tag{78}$$

and

$$\mathbf{Y}_t^{1,(x,h),\varpi,\mathbf{u}} = \big(X_t^{\#_{I_1^\varpi}+1,([(x,\mathbf{u})]_1^\varpi,h)}, X_t^{\#_{I_2^\varpi}+1,([(x,\mathbf{u})]_2^\varpi,h)}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi}+1,([(x,\mathbf{u})]_{\#_\varpi}^\varpi,h)}, X_t^{1,(x,h)}\big). \tag{79}$$

Next note that Lemma 3.1 (with $V = H$, $W = V$, $n = m$, $\varphi = \varphi$, $\Phi = \Phi_m$ for $m \in \{1, 2, \dots, k\}$ in the notation of Lemma 3.1) shows that for all $m \in \{1, 2, \dots, k\}$, $\mathbf{u} = (u_1, u_2, \dots, u_{m+1})$, $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{m+1}) \in H^{m+1}$ it holds that $\Phi_m \in C^1(H^{m+1}, V)$ and

$$\begin{aligned}\Phi_m'(\mathbf{u})\tilde{\mathbf{u}} &= \varphi^{(m+1)}(u_{m+1})(u_1, u_2, \dots, u_m, \tilde{u}_{m+1}) \\ &\quad + \textstyle\sum_{i=1}^m \varphi^{(m)}(u_{m+1})(u_1, u_2, \dots, u_{i-1}, \tilde{u}_i, u_{i+1}, u_{i+2}, \dots, u_m).\end{aligned} \tag{80}$$

This and Hölder's inequality imply that for all $m \in \{1, 2, \dots, k\}$, $\mathbf{u} = (u_1, u_2, \dots, u_{m+1})$, $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{m+1}) \in H^{m+1}$ it holds that

$$\begin{aligned}&\|\Phi_m'(\mathbf{u})\tilde{\mathbf{u}}\|_V \\ &\leq |\varphi|_{C_b^{m+1}(H,V)} \|\tilde{u}_{m+1}\|_H \textstyle\prod_{i=1}^m \|u_i\|_H + \sum_{i=1}^m |\varphi|_{C_b^m(H,V)} \|\tilde{u}_i\|_H \prod_{j\in\{1,2,\dots,m\}\setminus\{i\}} \|u_j\|_H \\ &\leq \|\tilde{\mathbf{u}}\|_{H^{m+1}} \Big[|\varphi|^2_{C_b^{m+1}(H,V)} \textstyle\prod_{i=1}^m \|u_i\|_H^2 + |\varphi|^2_{C_b^m(H,V)} \sum_{i=1}^m \prod_{j\in\{1,2,\dots,m\}\setminus\{i\}} \|u_j\|_H^2\Big]^{1/2}.\end{aligned} \tag{81}$$

Hence, we obtain that for all $m \in \{1, 2, \ldots, k\}$, $\mathbf{u} = (u_1, u_2, \ldots, u_{m+1}) \in H^{m+1}$ it holds that

$$\begin{aligned}
&\|\Phi'_m(\mathbf{u})\|_{L(H^{m+1},V)} \leq \|\varphi\|_{C_b^{m+1}(H,V)} \big[\textstyle\prod_{i=1}^{m} \|u_i\|_H^2 + \sum_{i=1}^{m} \prod_{j\in\{1,2,\ldots,m\}\setminus\{i\}} \|u_j\|_H^2\big]^{1/2} \\
&\leq \|\varphi\|_{C_b^{m+1}(H,V)} \\
&\quad \cdot \big[\textstyle\prod_{i=1}^{m} |\max\{1, \|\mathbf{u}\|_{H^{m+1}}\}|^2 + \sum_{i=1}^{m} \prod_{j\in\{1,2,\ldots,m\}\setminus\{i\}} |\max\{1, \|\mathbf{u}\|_{H^{m+1}}\}|^2\big]^{1/2} \\
&\leq \sqrt{m+1}\, \|\varphi\|_{C_b^{m+1}(H,V)}\, |\max\{1, \|\mathbf{u}\|_{H^{m+1}}\}|^m.
\end{aligned} \tag{82}$$

This shows that for all $m \in \{1, 2, \ldots, k\}$ it holds that

$$\sup_{\mathbf{u}\in H^{m+1}} \frac{\|\Phi'_m(\mathbf{u})\|_{L(H^{m+1},V)}}{|\max\{1, \|\mathbf{u}\|_{H^{m+1}}\}|^m} < \infty. \tag{83}$$

Next we note that for all $m \in \mathbb{N}$, $p \in [1, \infty)$, $Y_1, Y_2, \ldots, Y_m \in \mathcal{L}^0(\mathbb{P}; H)$ it holds that

$$\begin{aligned}
\|(Y_1, Y_2, \ldots, Y_m)\|_{\mathcal{L}^p(\mathbb{P};H^m)} &= \big\|[\textstyle\sum_{i=1}^{m} \|Y_i\|_H^2]^{1/2}\big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \leq \big\|\sum_{i=1}^{m} \|Y_i\|_H\big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \\
&\leq \textstyle\sum_{i=1}^{m} \|Y_i\|_{\mathcal{L}^p(\mathbb{P};H)}.
\end{aligned} \tag{84}$$

This shows that for all $m \in \{0, 1\}$, $p \in [1, \infty)$, $\varpi \in \Pi_k$, $\mathbf{u} \in H^k$, $\mathbf{v} \in H^{m+1}$, $t \in [0, T]$ it holds that

$$\mathbf{Y}_t^{m,\mathbf{v},\varpi,\mathbf{u}} \in \mathcal{L}^p(\mathbb{P}; H^{\#_\varpi+1}). \tag{85}$$

Next observe that (63), (84), and Jensen's inequality imply that for all $p \in [1, \infty)$, $\varpi \in \Pi_k$, $\mathbf{u} \in H^k$, $x \in H$, $t \in [0, T]$ it holds that

$$\big(H \ni h \mapsto [\mathbf{Y}_t^{1,(x,h),\varpi,\mathbf{u}}]_{\mathbb{P},\mathcal{B}(H^{\#_\varpi+1})} \in L^p(\mathbb{P}; H^{\#_\varpi+1})\big) \in L(H, L^p(\mathbb{P}; H^{\#_\varpi+1})). \tag{86}$$

Furthermore, we note that (51) and item (vi) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = 0$, $\beta = 0$, $k = m$, $p = p$, $x = x$ for $x \in H$, $p \in [2, \infty)$, $m \in \{2, 3, \ldots, k + 1\}$ in the notation of Theorem 2.1 in [2]) ensure that for all $m \in \{2, 3, \ldots, k + 1\}$, $p \in [2, \infty)$, $x \in H$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\limsup_{H\setminus\{0\}\ni u_m\to 0}\ \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_{m-1})\in(H\setminus\{0\})^{m-1}} \\
&\left[\frac{\|X_t^{m-1,(x+u_m,\mathbf{u})} - X_t^{m-1,(x,\mathbf{u})} - X_t^{m,(x,\mathbf{u},u_m)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^{m} \|u_i\|_H}\right] = 0.
\end{aligned} \tag{87}$$

Combining (75) and (87) with (84) and Jensen's inequality yields that for all $p \in [1, \infty)$, $\varpi \in \Pi_k$, $\mathbf{u} \in H^k$, $x \in H$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\limsup_{H\setminus\{0\}\ni h\to 0} \left[\frac{\big\|\mathbf{Y}_t^{0,x+h,\varpi,\mathbf{u}} - \mathbf{Y}_t^{0,x,\varpi,\mathbf{u}} - \mathbf{Y}_t^{1,(x,h),\varpi,\mathbf{u}}\big\|_{\mathcal{L}^p(\mathbb{P};H^{\#_\varpi+1})}}{\|h\|_H}\right] \\
&\leq \limsup_{H\setminus\{0\}\ni h\to 0} \left[\frac{\|X_t^{0,x+h} - X_t^{0,x} - X_t^{1,(x,h)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|h\|_H}\right] \\
&+ \sum_{i=1}^{\#_\varpi} \limsup_{H\setminus\{0\}\ni h\to 0} \left[\frac{\Big\|X_t^{\#_{I_i^\varpi},[(x+h,\mathbf{u})]_i^\varpi} - X_t^{\#_{I_i^\varpi},[(x,\mathbf{u})]_i^\varpi} - X_t^{\#_{I_i^\varpi}+1,([(x,\mathbf{u})]_i^\varpi,h)}\Big\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|h\|_H}\right] = 0.
\end{aligned} \tag{88}$$

In addition, combining (72) with (84) and Jensen's inequality yields that for all $p \in [1, \infty)$, $\varpi \in \Pi_k$, $\mathbf{u} \in H^k$, $x \in H$, $t \in [0, T]$ it holds that

$$
\begin{aligned}
&\limsup_{H \ni y \to x} \sup_{h \in H \setminus \{0\}} \left[\frac{\|\mathbf{Y}_t^{1,(x,h),\varpi,\mathbf{u}} - \mathbf{Y}_t^{1,(y,h),\varpi,\mathbf{u}}\|_{\mathcal{L}^p(\mathbb{P};H^{\#_\varpi+1})}}{\|h\|_H} \right] \\
&\leq \limsup_{H \ni y \to x} \sup_{h \in H \setminus \{0\}} \left[\frac{\|X_t^{1,(x,h)} - X_t^{1,(y,h)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|h\|_H} \right] \\
&+ \sum_{i=1}^{\#_\varpi} \limsup_{H \ni y \to x} \sup_{h \in H \setminus \{0\}} \left[\frac{\|X_t^{\#_{I_i^\varpi}+1,([x,\mathbf{u}]_i^\varpi,h)} - X_t^{\#_{I_i^\varpi}+1,([y,\mathbf{u}]_i^\varpi,h)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|h\|_H} \right] = 0.
\end{aligned}
\tag{89}
$$

Combining (86) and (88) hence yields that for all $p \in [1, \infty)$, $\varpi \in \Pi_k$, $\mathbf{u} \in H^k$, $x, h \in H$, $t \in [0, T]$ it holds that

$$
\big(H \ni y \mapsto [\mathbf{Y}_t^{0,y,\varpi,\mathbf{u}}]_{\mathbb{P},\mathcal{B}(H^{\#_\varpi+1})} \in L^p(\mathbb{P}; H^{\#_\varpi+1})\big) \in C^1(H, L^p(\mathbb{P}; H^{\#_\varpi+1})) \tag{90}
$$

and

$$
\big(\tfrac{\partial}{\partial x}[\mathbf{Y}_t^{0,x,\varpi,\mathbf{u}}]_{\mathbb{P},\mathcal{B}(H^{\#_\varpi+1})}\big)h = [\mathbf{Y}_t^{1,(x,h),\varpi,\mathbf{u}}]_{\mathbb{P},\mathcal{B}(H^{\#_\varpi+1})}. \tag{91}
$$

This, (80), (83), and Lemma 2.1 (with $U = H$, $V = H^{\#_\varpi+1}$, $W = V$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $X^{0,x} = \mathbf{Y}_t^{0,x,\varpi,\mathbf{u}}$, $X^{1,(x,h)} = \mathbf{Y}_t^{1,(x,h),\varpi,\mathbf{u}}$, $\varphi = \Phi_{\#_\varpi}$ for $t \in [0, T]$, $x, h \in H$, $\mathbf{u} \in H^k$, $\varpi \in \Pi_k$ in the notation of Lemma 2.1) assure that

(a) it holds for all $\varpi \in \Pi_k$, $\mathbf{u} \in H^k$, $t \in [0, T]$ that

$$
\begin{aligned}
&\Big(H \ni x \mapsto \mathbb{E}\Big[\Phi_{\#_\varpi}\big(\mathbf{Y}_t^{0,x,\varpi,\mathbf{u}}\big)\Big] \\
&= \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big] \in V\Big) \in C^1(H, V)
\end{aligned}
\tag{92}
$$

and

(b) it holds for all $\varpi \in \Pi_k$, $\mathbf{u} \in H^k$, $x, u_{k+1} \in H$, $t \in [0, T]$ that

$$
\begin{aligned}
&\Big(\tfrac{\partial}{\partial x}\mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big]\Big)u_{k+1} \\
&= \Big(\tfrac{\partial}{\partial x}\mathbb{E}\Big[\Phi_{\#_\varpi}\big(\mathbf{Y}_t^{0,x,\varpi,\mathbf{u}}\big)\Big]\Big)u_{k+1} \\
&= \mathbb{E}\Big[\Phi'_{\#_\varpi}\big(\mathbf{Y}_t^{0,x,\varpi,\mathbf{u}}\big)\mathbf{Y}_t^{1,(x,u_{k+1}),\varpi,\mathbf{u}}\Big] \\
&= \mathbb{E}\Big[\varphi^{(\#_\varpi+1)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{1,(x,u_{k+1})}\big)\Big] \\
&\quad + \textstyle\sum_{i=1}^{\#_\varpi} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi},[(x,\mathbf{u})]_{i-1}^\varpi}, \\
&\quad X_t^{\#_{I_i^\varpi}+1,([(x,\mathbf{u})]_i^\varpi,u_{k+1})}, X_t^{\#_{I_{i+1}^\varpi},[(x,\mathbf{u})]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi},[(x,\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big].
\end{aligned}
\tag{93}
$$

Combining item (a) with the induction hypothesis shows that for all $\mathbf{u} \in H^k$, $t \in [0, T]$ it holds that

$$\Big(H \ni x \mapsto \sum_{\varpi \in \Pi_k} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big] = \big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x)\mathbf{u} \in V\Big) \in C^1(H, V). \quad (94)$$

Item (b) hence proves that for all $\mathbf{u} \in H^k$, $x, h \in H$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\big(\tfrac{d}{dx}\big[\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x)\mathbf{u}\big]\big)h \\
&= \sum_{\varpi \in \Pi_k} \Big\{\mathbb{E}\Big[\varphi^{(\#_\varpi+1)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{1,(x,h)}\big)\Big] \\
&\quad + \textstyle\sum_{i=1}^{\#_\varpi} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I_{i-1}^\varpi},[(x,\mathbf{u})]_{i-1}^\varpi}, \\
&\quad X_t^{\#_{I_i^\varpi}+1,([(x,\mathbf{u})]_i^\varpi,h)}, X_t^{\#_{I_{i+1}^\varpi},[(x,\mathbf{u})]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi},[(x,\mathbf{u})]_{i+2}^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big]\Big\}.
\end{aligned} \quad (95)$$

In addition, note that

$$\begin{aligned}
\Pi_{k+1} = &\Big\{\varpi \cup \big\{\{k+1\}\big\} : \varpi \in \Pi_k\Big\} \\
&\uplus \Big\{\big\{I_1^\varpi, I_2^\varpi, \dots, I_{i-1}^\varpi, I_i^\varpi \cup \{k+1\}, I_{i+1}^\varpi, I_{i+2}^\varpi, \dots, I_{\#_\varpi}^\varpi\big\} : i \in \{1, 2, \dots, \#_\varpi\},\ \varpi \in \Pi_k\Big\}.
\end{aligned} \quad (96)$$

This ensures that for all $\mathbf{u} = (u_0, u_1, \dots, u_k) \in H^{k+1}$, $h \in H$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\textstyle\sum_{\varpi \in \Pi_{k+1}} \varphi^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[(\mathbf{u},h)]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(\mathbf{u},h)]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(\mathbf{u},h)]_{\#_\varpi}^\varpi}\big) \\
&= \textstyle\sum_{\varpi \in \Pi_k} \Big[\varphi^{(\#_\varpi+1)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}, X_t^{1,(u_0,h)}\big) \\
&\quad + \textstyle\sum_{i=1}^{\#_\varpi} \varphi^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \dots, X_t^{\#_{I_{i-1}^\varpi},[\mathbf{u}]_{i-i}^\varpi}, X_t^{\#_{I_i^\varpi}+1,([\mathbf{u}]_i^\varpi,h)}, \\
&\quad X_t^{\#_{I_{i+1}^\varpi},[\mathbf{u}]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi},[\mathbf{u}]_{i+2}^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\Big].
\end{aligned} \quad (97)$$

Combining this with (95) establishes that for all $\mathbf{u} \in H^k$, $x, h \in H$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\big(\tfrac{d}{dx}\big[\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x)\mathbf{u}\big]\big)h \\
&= \sum_{\varpi \in \Pi_{k+1}} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u},h)]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u},h)]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u},h)]_{\#_\varpi}^\varpi},\big)\Big].
\end{aligned} \quad (98)$$

Hence, we obtain that for all $\mathbf{u} = (u_1, u_2, \dots, u_k) \in H^k$, $x \in H$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\limsup_{H\setminus\{0\}\ni h\to 0} \bigg(\frac{1}{\|h\|_H}\Big\|\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x+h)\mathbf{u} - \big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x)\mathbf{u} \\
&- \textstyle\sum_{\varpi \in \Pi_{k+1}} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u},h)]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u},h)]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u},h)]_{\#_\varpi}^\varpi}\big)\Big]\Big\|_V\bigg) = 0.
\end{aligned} \quad (99)$$

Combining (65) and Lemma 2.2 (with $V = H$, $W = V$, $n = k$, $f = (H \ni x \mapsto \phi(t,x) \in V)$, $g = (H \ni x \mapsto (H^{k+1} \ni \mathbf{u} \mapsto \sum_{\varpi \in \Pi_{k+1}} \mathbb{E}[\varphi^{(\#_\varpi)}(X_t^{0,x})(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi})] \in V) \in L^{(k+1)}(H,V))$ for $t \in [0,T]$ in the notation of Lemma 2.2) therefore shows that for all $\mathbf{u} \in H^{k+1}$, $x \in H$, $t \in [0,T]$ it holds that $(H \ni y \mapsto \phi(t,y) \in V) \in C^{k+1}(H,V)$ and

$$\begin{aligned}&\left(\tfrac{\partial^{k+1}}{\partial x^{k+1}}\phi\right)(t,x)\mathbf{u}\\ &= \sum_{\varpi \in \Pi_{k+1}} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big].\end{aligned} \tag{100}$$

This and (62) prove (74) in the case $k+1$. Induction thus completes the proof of (74).

Next observe that item (ii) and item (iii) follow immediately from (74). It thus remains to prove items (iv)–(viii). To prove item (iv) we first note that (51) and item (ii) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = \alpha$, $\beta = \beta$, $k = k$, $p = p$, $\boldsymbol{\delta} = \boldsymbol{\delta}$ for $\boldsymbol{\delta} \in \mathbb{D}_k$, $p \in [2,\infty)$, $k \in \{1,2,\dots,n\}$ in the notation of Theorem 2.1 in [2]) ensure that for all $k \in \{1,2,\dots,n\}$, $p \in [2,\infty)$, $\boldsymbol{\delta} = (\delta_1,\delta_2,\dots,\delta_k) \in \mathbb{D}_k$ it holds that

$$\sup_{\mathbf{u}=(u_0,u_1,\dots,u_k) \in H \times (H\setminus\{0\})^k} \sup_{t \in (0,T]} \left[\frac{t^{\iota_{\mathbb{N}}^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{k,\mathbf{u}}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right] < \infty. \tag{101}$$

This and Jensen's inequality establish item (iv). Moreover, observe that for all $k \in \mathbb{N}$, $\boldsymbol{\delta} = (\delta_1,\delta_2,\dots,\delta_k) \in \mathbb{R}^k$, $\varpi \in \Pi_k$ it holds that

$$\begin{aligned}\sup_{t\in(0,T]} \prod_{I \in \varpi} t^{(-\iota_I^{\boldsymbol{\delta},\alpha,\beta}+\sum_{i\in I}\delta_i)} &= \sup_{t\in(0,T]} t^{\min\{1-\alpha,1/2-\beta\}\sum_{I\in\varpi}\mathbb{1}_{[2,\infty)}(\#_I)}\\ &= T^{\min\{1-\alpha,1/2-\beta\}\sum_{I\in\varpi}\mathbb{1}_{[2,\infty)}(\#_I)} \le |T \vee 1|^{\lfloor k/2\rfloor \min\{1-\alpha,1/2-\beta\}}.\end{aligned} \tag{102}$$

Combining (60) with item (iii), (101), (102), and Jensen's inequality yields that for all $k \in \{1,2,\dots,n\}$, $\boldsymbol{\delta} = (\delta_1,\delta_2,\dots,\delta_k) \in \mathbb{D}_k$ it holds that

$$\begin{aligned}&\sup_{v\in H} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \left[\frac{t^{\sum_{i=1}^k \delta_i} \left\|\left(\frac{\partial^k}{\partial x^k}\phi\right)(t,v)\mathbf{u}\right\|_V}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right]\\ &\le \sum_{\varpi\in\Pi_k} |\varphi|_{C_b^{\#_\varpi}(H,V)} \left(\sup_{t\in(0,T]}\left[\prod_{I\in\varpi} t^{(-\iota_I^{\boldsymbol{\delta},\alpha,\beta}+\sum_{i\in I}\delta_i)}\right]\right)\\ &\quad \cdot \left(\prod_{I\in\varpi} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t\in(0,T]}\left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right]\right)\\ &\le |T\vee 1|^{\lfloor k/2\rfloor\min\{1-\alpha,1/2-\beta\}} \|\varphi\|_{C_b^k(H,V)}\\ &\quad \cdot \sum_{\varpi\in\Pi_k}\prod_{I\in\varpi} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t\in(0,T]}\left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right] < \infty.\end{aligned} \tag{103}$$

This proves item (v). Next we observe that (51) and item (iv) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = \alpha$, $\beta = \beta$, $k = k$, $p = p$, $\boldsymbol{\delta} = \boldsymbol{\delta}$ for $\boldsymbol{\delta} \in \mathbb{D}_k$, $p \in [2,\infty)$,

$k \in \{l \in \{1, 2, \ldots, n\} \colon |F|_{\mathrm{Lip}^l(H, H_{-\alpha})} + |B|_{\mathrm{Lip}^l(H, HS(U, H_{-\beta}))} < \infty\}$ in the notation of Theorem 2.1 in [2]) ensure that for all $k \in \{1, 2, \ldots, n\}$, $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{D}_k$ with $|F|_{\mathrm{Lip}^k(H, H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H, HS(U, H_{-\beta}))} < \infty$ it holds that

$$\sup_{\substack{x, y \in H,\\ x \neq y}} \sup_{\mathbf{u} = (u_1, u_2, \ldots, u_k) \in (H \setminus \{0\})^k} \sup_{t \in (0, T]} \left[\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta}, 0), \alpha, \beta}} \| X_t^{k, (x, \mathbf{u})} - X_t^{k, (y, \mathbf{u})} \|_{\mathcal{L}^p(\mathbb{P}; H)}}{\|x - y\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right] < \infty. \tag{104}$$

Combining this and (67) with Jensen's inequality establish items (vi) and (vii). Moreover, note that for all $k \in \mathbb{N}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{R}^k$, $\varpi \in \Pi_k$, $I \in \varpi$ it holds that

$$\begin{aligned} &\sup_{t \in (0, T]} \left[t^{(-\iota_{I \cup \{k+1\}}^{(\boldsymbol{\delta}, 0), \alpha, \beta} + \sum_{i \in I} \delta_i)} \prod_{J \in \varpi \setminus \{I\}} t^{(-\iota_J^{\boldsymbol{\delta}, \alpha, \beta} + \sum_{i \in J} \delta_i)} \right] \\ &= \sup_{t \in (0, T]} t^{\min\{1 - \alpha, 1/2 - \beta\} \, [1 + \sum_{J \in \varpi \setminus \{I\}} \mathbb{1}_{[2, \infty)}(\#_J)]} \\ &= T^{\min\{1 - \alpha, 1/2 - \beta\} \, [1 + \sum_{J \in \varpi \setminus \{I\}} \mathbb{1}_{[2, \infty)}(\#_J)]} \leq |T \vee 1|^{\lceil k/2 \rceil \min\{1 - \alpha, 1/2 - \beta\}}. \end{aligned} \tag{105}$$

Furthermore, note that item (iii) and (66) imply that for all $k \in \{1, 2, \ldots, n\}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{D}_k$ with $|\varphi|_{\mathrm{Lip}^k(H, V)} < \infty$ it holds that

$$\begin{aligned} &\sup_{\substack{v, w \in H,\\ v \neq w}} \sup_{\mathbf{u} = (u_1, u_2, \ldots, u_k) \in (H \setminus \{0\})^k} \sup_{t \in (0, T]} \left[\frac{t^{\sum_{i=1}^k \delta_i} \left\| \left[\left(\tfrac{\partial^k}{\partial x^k} \phi \right)(t, v) - \left(\tfrac{\partial^k}{\partial x^k} \phi \right)(t, w) \right] \mathbf{u} \right\|_V}{\|v - w\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right] \\ &\leq \sum_{\varpi \in \Pi_k} \Bigg\{ |\varphi|_{\mathrm{Lip}^{\#_\varpi}(H, V)} \left(\sup_{t \in (0, T]} \left[\prod_{I \in \varpi} t^{(-\iota_I^{\boldsymbol{\delta}, \alpha, \beta} + \sum_{i \in I} \delta_i)} \right] \right) \\ &\cdot \left(\sup_{\substack{x, y \in H,\\ x \neq y}} \sup_{t \in (0, T]} \left[\frac{\| X_t^{0, x} - X_t^{0, y} \|_{\mathcal{L}^{\#_\varpi + 1}(\mathbb{P}; H)}}{\|x - y\|_H} \right] \right) \\ &\cdot \left(\prod_{I \in \varpi} \sup_{x \in H} \sup_{\mathbf{u} = (u_i)_{i \in I} \in (H \setminus \{0\})^{\#_I}} \sup_{t \in (0, T]} \left[\frac{t^{\iota_I^{\boldsymbol{\delta}, \alpha, \beta}} \| X_t^{\#_I, (x, \mathbf{u})} \|_{\mathcal{L}^{\#_\varpi + 1}(\mathbb{P}; H)}}{\prod_{i \in I} \|u_i\|_{H_{-\delta_i}}} \right] \right) \\ &+ |\varphi|_{C_b^{\#_\varpi}(H, V)} \sum_{I \in \varpi} \left(\sup_{t \in (0, T]} \left[t^{(-\iota_{I \cup \{k+1\}}^{(\boldsymbol{\delta}, 0), \alpha, \beta} + \sum_{i \in I} \delta_i)} \prod_{J \in \varpi \setminus \{I\}} t^{(-\iota_J^{\boldsymbol{\delta}, \alpha, \beta} + \sum_{i \in J} \delta_i)} \right] \right) \\ &\cdot \left(\sup_{\substack{x, y \in H,\\ x \neq y}} \sup_{\mathbf{u} = (u_i)_{i \in I} \in (H \setminus \{0\})^{\#_I}} \sup_{t \in (0, T]} \left[\frac{t^{\iota_{I \cup \{k+1\}}^{(\boldsymbol{\delta}, 0), \alpha, \beta}} \| X_t^{\#_I, (x, \mathbf{u})} - X_t^{\#_I, (y, \mathbf{u})} \|_{\mathcal{L}^{\#_\varpi}(\mathbb{P}; H)}}{\|x - y\|_H \prod_{i \in I} \|u_i\|_{H_{-\delta_i}}} \right] \right) \\ &\cdot \left(\prod_{J \in \varpi \setminus \{I\}} \sup_{x \in H} \sup_{\mathbf{u} = (u_i)_{i \in J} \in (H \setminus \{0\})^{\#_J}} \sup_{t \in (0, T]} \left[\frac{t^{\iota_J^{\boldsymbol{\delta}, \alpha, \beta}} \| X_t^{\#_J, (x, \mathbf{u})} \|_{\mathcal{L}^{\#_\varpi}(\mathbb{P}; H)}}{\prod_{i \in J} \|u_i\|_{H_{-\delta_i}}} \right] \right) \Bigg\}. \end{aligned} \tag{106}$$

Combining (106) with (67), (101), (102), (104), (105), and Jensen's inequality establishes item (viii). The proof of Lemma 3.2 is thus completed. □

Theorem 3.3 *Assume the setting in* Section 1.2 *and let $n \in \mathbb{N}$, $\varphi \in C_b^n(H, V)$, $F \in C_b^n(H, H)$, $B \in C_b^n(H, HS(U, H))$. Then*

(i) *it holds that there exist up-to-modifications unique* $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$*-predictable stochastic processes* $X^{k,\mathbf{u}}\colon [0,T]\times\Omega\to H$, $\mathbf{u}\in H^{k+1}$, $k\in\{0,1,\dots,n\}$, *which satisfy for all* $k\in\{0,1,\dots,n\}$, $\mathbf{u}=(u_0,u_1,\dots,u_k)\in H^{k+1}$, $p\in(0,\infty)$, $t\in[0,T]$ *that* $\sup_{s\in[0,T]}\mathbb{E}\big[\|X_s^{k,\mathbf{u}}\|_H^p\big]<\infty$ *and*

$$\begin{aligned}
&[X_t^{k,\mathbf{u}} - e^{tA}\,\mathbb{1}_{\{0,1\}}(k)\,u_k]_{\mathbb{P},\mathcal{B}(H)}\\
&=\int_0^t e^{(t-s)A}\bigg[\mathbb{1}_{\{0\}}(k)\,F(X_s^{0,u_0})\\
&\quad+\sum_{\varpi\in\Pi_k} F^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\bigg]\,\boldsymbol{ds}\\
&\quad+\int_0^t e^{(t-s)A}\bigg[\mathbb{1}_{\{0\}}(k)\,B(X_s^{0,u_0})\\
&\quad+\sum_{\varpi\in\Pi_k} B^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\bigg]\,dW_s,
\end{aligned}\tag{107}$$

(ii) *it holds that there exists a unique function* $\phi\colon[0,T]\times H\to V$ *which satisfies for all* $t\in[0,T]$, $x\in H$ *that* $\phi(t,x)=\mathbb{E}[\varphi(X_t^{0,x})]$,

(iii) *it holds for all* $t\in[0,T]$ *that* $\big(H\ni x\mapsto\phi(t,x)\in V\big)\in C_b^n(H,V)$,

(iv) *it holds for all* $k\in\{1,2,\dots,n\}$, $\mathbf{u}=(u_0,u_1,\dots,u_k)\in H^{k+1}$, $t\in[0,T]$ *that*

$$\sum_{\varpi\in\Pi_k}\mathbb{E}\Big[\big\|\varphi^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\big\|_V\Big]<\infty,\tag{108}$$

(v) *it holds for all* $k\in\{1,2,\dots,n\}$, $\mathbf{u}\in H^k$, $x\in H$, $t\in[0,T]$ *that*

$$\begin{aligned}
&\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x)\mathbf{u}\\
&=\sum_{\varpi\in\Pi_k}\mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[(x,\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[(x,\mathbf{u})]_2^\varpi},\dots,X_t^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big],
\end{aligned}\tag{109}$$

(vi) *it holds for all* $p\in(0,\infty)$, $k\in\{1,2,\dots,n\}$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\dots,\delta_k)\in[0,1/2)^k$, $\alpha\in[0,1)$, $\beta\in[0,1/2)$ *with* $\sum_{i=1}^k\delta_i<1/2$ *that*

$$\sup_{x\in H}\;\sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\backslash\{0\})^k}\;\sup_{t\in(0,T]}\left[\frac{t^{\iota_{\mathbb{N}}^{\boldsymbol{\delta},\alpha,\beta}}\,\|X_t^{k,(x,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right]<\infty,\tag{110}$$

(vii) *it holds for all* $k\in\{1,2,\dots,n\}$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\dots,\delta_k)\in[0,1/2)^k$, $\alpha\in[0,1)$, $\beta\in[0,1/2)$ *with* $\sum_{i=1}^k\delta_i<1/2$ *that*

$$\begin{aligned}
&\sup_{v\in H}\;\sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\backslash\{0\})^k}\;\sup_{t\in(0,T]}\left[\frac{t^{\sum_{i=1}^k\delta_i}\,\big\|\big(\frac{\partial^k}{\partial x^k}\phi\big)(t,v)\mathbf{u}\big\|_V}{\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right]\\
&\leq\;|T\vee 1|^{\lfloor k/2\rfloor\,\min\{1-\alpha,1/2-\beta\}}\,\|\varphi\|_{C_b^k(H,V)}\\
&\quad\cdot\sum_{\varpi\in\Pi_k}\prod_{I\in\varpi}\sup_{x\in H}\;\sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\backslash\{0\})^{\#_I}}\;\sup_{t\in(0,T]}\left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}}\,\|X_t^{\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right]<\infty,
\end{aligned}\tag{111}$$

(viii) *it holds for all* $p \in (0, \infty)$ *that*

$$\sup_{\substack{x,y \in H,\\ x \neq y}} \sup_{t \in (0,T]} \left[\frac{\|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H} \right] < \infty, \tag{112}$$

(ix) *it holds for all* $p \in (0, \infty)$, $k \in \{1, 2, \ldots, n\}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in [0, 1/2)^k$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *and* $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} < \infty$ *that*

$$\sup_{\substack{x,y \in H,\\ x \neq y}} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \sup_{t \in (0,T]} \left[\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right] < \infty, \tag{113}$$

and

(x) *it holds for all* $k \in \{1, 2, \ldots, n\}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in [0, 1/2)^k$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *and* $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} + |\varphi|_{\mathrm{Lip}^k(H,V)} < \infty$ *that*

$$\begin{aligned}
&\sup_{\substack{v,w \in H,\\ v \neq w}} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \sup_{t \in (0,T]} \left[\frac{t^{\sum_{i=1}^k \delta_i} \left\| \left[\left(\frac{\partial^k}{\partial x^k}\phi\right)(t,v) - \left(\frac{\partial^k}{\partial x^k}\phi\right)(t,w) \right] \mathbf{u} \right\|_V}{\|v-w\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right] \\
&\leq |T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}} \|\varphi\|_{\mathrm{Lip}^k(H,V)} \\
&\cdot \sum_{\varpi \in \Pi_k} \left\{ \sup_{\substack{x,y \in H,\\ x \neq y}} \sup_{t \in (0,T]} \left[\frac{\|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^{\#\varpi+1}(\mathbb{P};H)}}{\|x-y\|_H} \right] \right. \\
&\cdot \prod_{I \in \varpi} \sup_{x \in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t \in (0,T]} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#\varpi+1}(\mathbb{P};H)}}{\prod_{i \in I} \|u_i\|_{H_{-\delta_i}}} \right] \\
&+ \sum_{I \in \varpi} \sup_{\substack{x,y \in H,\\ x \neq y}} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t \in (0,T]} \left[\frac{t^{\iota_{I\cup\{k+1\}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{\#_I,(x,\mathbf{u})} - X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{\#\varpi}(\mathbb{P};H)}}{\|x-y\|_H \prod_{i \in I} \|u_i\|_{H_{-\delta_i}}} \right] \\
&\left. \cdot \prod_{J \in \varpi\setminus\{I\}} \sup_{x \in H} \sup_{\mathbf{u}=(u_i)_{i\in J}\in(H\setminus\{0\})^{\#_J}} \sup_{t \in (0,T]} \left[\frac{t^{\iota_J^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\#_J,(x,\mathbf{u})}\|_{\mathcal{L}^{\#\varpi}(\mathbb{P};H)}}{\prod_{i \in J} \|u_i\|_{H_{-\delta_i}}} \right] \right\} < \infty.
\end{aligned} \tag{114}$$

Proof Note that items (i) and (ii) follow immediately from item (i) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = 0$, $\beta = 0$ in the notation of Theorem 2.1 in [2]). Moreover, observe that items (iii)–(x) follow directly from items (i)–(viii) of Lemma 3.2. The proof of Theorem 3.3 is thus completed. □

4 Regularity of Transition Semigroups for Mollified Stochastic Evolution Equations

Lemma 4.1 *Assume the setting in* Section 1.2, *let* $n \in \mathbb{N}$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$, $F \in C_b^n(H, H_{-\alpha})$, $B \in C_b^n(H, HS(U, H_{-\beta}))$, $\varphi \in C_b^n(H, V)$, *let* $X^{\varepsilon,k,\mathbf{u}}: [0, T] \times \Omega \to H$, $\mathbf{u} \in$

H^{k+1}, $k \in \{0,1,\dots,n\}$, $\varepsilon \in (0,T]$, *be* $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$*-predictable stochastic processes which satisfy for all* $k \in \{0,1,\dots,n\}$, $\mathbf{u} = (u_0,u_1,\dots,u_k) \in H^{k+1}$, $\varepsilon \in (0,T]$, $p \in (0,\infty)$, $t \in [0,T]$ *that* $\sup_{s\in[0,T]} \mathbb{E}\big[\|X_s^{\varepsilon,k,\mathbf{u}}\|_H^p\big] < \infty$ *and*

$$\begin{aligned}
&[X_t^{\varepsilon,k,\mathbf{u}} - e^{tA}\,\mathbb{1}_{\{0,1\}}(k)\,u_k]_{\mathbb{P},\mathcal{B}(H)}\\
&=\int_0^t e^{(t-s+\varepsilon)A}\Bigg[\mathbb{1}_{\{0\}}(k)\,F(X_s^{\varepsilon,0,u_0})\\
&\quad+\sum_{\varpi\in\Pi_k} F^{(\#_\varpi)}(X_s^{\varepsilon,0,u_0})\big(X_s^{\varepsilon,\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\varepsilon,\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_s^{\varepsilon,\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\Bigg]\,ds\\
&\quad+\int_0^t e^{(t-s+\varepsilon)A}\Bigg[\mathbb{1}_{\{0\}}(k)\,B(X_s^{\varepsilon,0,u_0})\\
&\quad+\sum_{\varpi\in\Pi_k} B^{(\#_\varpi)}(X_s^{\varepsilon,0,u_0})\big(X_s^{\varepsilon,\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\varepsilon,\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_s^{\varepsilon,\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\Bigg]\,dW_s,
\end{aligned} \tag{115}$$

and let $\phi_\varepsilon\colon [0,T]\times H \to V$, $\varepsilon \in (0,T]$, *be the functions which satisfy for all* $\varepsilon \in (0,T]$, $t \in [0,T]$, $x \in H$ *that* $\phi_\varepsilon(t,x) = \mathbb{E}[\varphi(X_t^{\varepsilon,0,x})]$. *Then*

(i) *it holds for all* $\varepsilon \in (0,T]$, $t \in [0,T]$ *that* $\big(H \ni x \mapsto \phi_\varepsilon(t,x) \in V\big) \in C_b^n(H,V)$,

(ii) *it holds for all* $k \in \{1,2,\dots,n\}$, $\delta_1,\delta_2,\dots,\delta_k \in [0,1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *that*

$$\sup_{\varepsilon,t\in(0,T]}\sup_{v\in H}\sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k}\left[\frac{t^{\sum_{i=1}^k\delta_i}\,\big\|\big(\tfrac{\partial^k}{\partial x^k}\phi_\varepsilon\big)(t,v)\mathbf{u}\big\|_V}{\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right] < \infty, \tag{116}$$

and

(iii) *it holds for all* $k \in \{1,2,\dots,n\}$, $\delta_1,\delta_2,\dots,\delta_k \in [0,1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *and* $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} + |\varphi|_{\mathrm{Lip}^k(H,V)} < \infty$ *that*

$$\begin{aligned}
&\sup_{\varepsilon,t\in(0,T]}\sup_{\substack{v,w\in H,\\ v\neq w}}\sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k}\\
&\left[\frac{t^{\sum_{i=1}^k\delta_i}\,\big\|\big[\big(\tfrac{\partial^k}{\partial x^k}\phi_\varepsilon\big)(t,v)-\big(\tfrac{\partial^k}{\partial x^k}\phi_\varepsilon\big)(t,w)\big]\mathbf{u}\big\|_V}{\|v-w\|_H\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right] < \infty.
\end{aligned} \tag{117}$$

Proof Throughout this proof let $\mathbb{D}_k \in \mathcal{P}(\mathbb{R}^k)$, $k \in \mathbb{N}$, be the sets which satisfy for all $k \in \mathbb{N}$ that $\mathbb{D}_k = \{(\delta_1,\delta_2,\dots,\delta_k) \in [0,1/2)^k\colon \sum_{i=1}^k \delta_i < 1/2\}$, let $\chi_r \in [0,\infty)$, $r \in [0,1]$, be the real numbers which satisfy for all $r \in [0,1]$ that $\chi_r = \sup_{t\in(0,T]} t^r\,\|(\eta-A)^r e^{tA}\|_{L(H)}$ (see, e.g., Lemma 11.36 in Renardy & Rogers [20]), let $\mathbb{B}\colon (0,\infty)^2 \to (0,\infty)$ be the function which satisfies for all $x,y \in (0,\infty)$ that $\mathbb{B}(x,y) = \int_0^1 t^{(x-1)}\,(1-t)^{(y-1)}\,dt$ (Beta function), let $\mathrm{E}_{a,b}\colon [0,\infty) \to [0,\infty)$, $a,b \in (-\infty,1)$, be the functions which satisfy for all $a,b \in (-\infty,1)$, $x \in [0,\infty)$ that $\mathrm{E}_{a,b}[x] = 1 + \sum_{n=1}^\infty x^n \prod_{k=0}^{n-1}\mathbb{B}\big(1-b, k(1-b)+1-a\big)$ (generalized exponential function; cf. Chapter 7 in Henry [15] and (16) in [1]), let $F_\varepsilon\colon H \to H$, $\varepsilon \in (0,T]$, and $B_\varepsilon\colon H \to HS(U,H)$, $\varepsilon \in (0,T]$, be the functions which satisfy for all $\varepsilon \in (0,T]$, $x \in H$, $u \in U$ that

$$F_\varepsilon(x) = e^{\varepsilon A}F(x) \qquad \text{and} \qquad B_\varepsilon(x)u = e^{\varepsilon A}B(x)u, \tag{118}$$

and let $\Theta_p^\lambda\colon [0,\infty)^2 \to [0,\infty]$, $p \in [1,\infty)$, $\lambda \in (-\infty,1)$, and $\Upsilon_p^\lambda\colon [0,\infty)^2 \to [0,\infty]$, $p \in [1,\infty)$, $\lambda \in (-\infty,1)$, be the functions which satisfy for all $\lambda \in (-\infty,1)$, $p \in [1,\infty)$, $L, \hat{L} \in [0,\infty)$ that

$$\Theta_p^\lambda(L,\hat{L}) = \begin{cases} \sqrt{2}\left|E_{2\lambda,\max\{\alpha,2\beta\}}\left[\left|\frac{\chi_\alpha\, L\, \sqrt{2}\, T^{(1-\alpha)}}{\sqrt{1-\alpha}} + \chi_\beta\, \hat{L}\, \sqrt{p\,(p-1)\, T^{(1-2\beta)}}\right|^2\right]\right|^{1/2} &: (\lambda,\hat{L}) \in \left(-\infty,\frac{1}{2}\right)\times(0,\infty) \\ E_{\lambda,\alpha}\left[\chi_\alpha\, L\, T^{(1-\alpha)}\right] &: \hat{L} = 0 \\ \infty &: \text{otherwise} \end{cases} \tag{119}$$

(see, e.g., (17) in [1]) and

$$\Upsilon_p^\lambda(L,\hat{L}) = \sup_{x\in[0,L]}\, \sup_{y\in[0,\hat{L}]} \Theta_p^\lambda(x,y). \tag{120}$$

Note that for all $\lambda \in (-\infty, 1/2)$, $p \in [1,\infty)$, $L, \hat{L} \in [0,\infty)$ it holds that

$$\Upsilon_p^\lambda(L,\hat{L}) = \max\left\{\Theta_p^\lambda(L,\hat{L}), \Theta_p^\lambda(L,0)\right\} < \infty. \tag{121}$$

Moreover, observe that for all $\varepsilon \in (0,T]$ it holds that

$$F_\varepsilon \in C_b^n(H,H) \qquad \text{and} \qquad B_\varepsilon \in C_b^n(H, HS(U,H)). \tag{122}$$

In addition, note that for all $k \in \{1,2,\ldots,n\}$, $\varepsilon \in (0,T]$ it holds that

$$\begin{aligned} &|F_\varepsilon|_{C_b^k(H,H_{-\alpha})} \leq \chi_0\, |F|_{C_b^k(H,H_{-\alpha})} < \infty \qquad \text{and} \\ &|B_\varepsilon|_{C_b^k(H,HS(U,H_{-\beta}))} \leq \chi_0\, |B|_{C_b^k(H,HS(U,H_{-\beta}))} < \infty. \end{aligned} \tag{123}$$

Furthermore, note that for all $k \in \{0,1,\ldots,n\}$, $\varepsilon \in (0,T]$ it holds that

$$\begin{aligned} &|F_\varepsilon|_{\mathrm{Lip}^k(H,H_{-\alpha})} \leq \chi_0\, |F|_{\mathrm{Lip}^k(H,H_{-\alpha})} \qquad \text{and} \\ &|B_\varepsilon|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} \leq \chi_0\, |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))}. \end{aligned} \tag{124}$$

Item (ii) of Lemma 3.2 (with $n = n$, $\varphi = \varphi$, $F = F_\varepsilon$, $B = B_\varepsilon$, $X^{k,\mathbf{u}} = X^{\varepsilon,k,\mathbf{u}}$, $\phi = \phi_\varepsilon$ $t = t$ for $t \in [0,T]$, $\varepsilon \in (0,T]$, $\mathbf{u} \in H^{k+1}$, $k \in \{0,1,\ldots,n\}$ in the notation of Lemma 3.2) and (122) prove item (i). Next we combine (115) and item (iii) of Corollary 2.10 in [1] (with $H = H, U = U, T = T, \eta = \eta, \alpha = \alpha, \beta = \beta, W = W, A = A, F = (H \ni x \mapsto F_\varepsilon(x) \in H_{-\alpha})$, $B = (H \ni x \mapsto (U \ni u \mapsto B_\varepsilon(x)u \in H_{-\beta}) \in HS(U, H_{-\beta}))$, $p = p$, $\delta = 0$ for $\varepsilon \in (0,T]$, $p \in [2,\infty)$ in the notation of Corollary 2.10 in [1]) with (121) and (124) to obtain that for all $p \in [2,\infty)$ it holds that

$$\begin{aligned} &\sup_{\varepsilon,t\in(0,T]}\, \sup_{\substack{x,y\in H,\\ x\neq y}} \left[\frac{\|X_t^{\varepsilon,0,x} - X_t^{\varepsilon,0,y}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H}\right] \\ &\leq \chi_0 \sup_{\varepsilon\in(0,T]} \Theta_p^0\big(|F_\varepsilon|_{\mathrm{Lip}^0(H,H_{-\alpha})}, |B_\varepsilon|_{\mathrm{Lip}^0(H,HS(U,H_{-\beta}))}\big) \\ &\leq \chi_0 \sup_{\varepsilon\in(0,T]} \Upsilon_p^0\big(|F_\varepsilon|_{\mathrm{Lip}^0(H,H_{-\alpha})}, |B_\varepsilon|_{\mathrm{Lip}^0(H,HS(U,H_{-\beta}))}\big) \\ &\leq \chi_0\, \Upsilon_p^0\big(\chi_0\, |F|_{\mathrm{Lip}^0(H,H_{-\alpha})}, \chi_0\, |B|_{\mathrm{Lip}^0(H,HS(U,H_{-\beta}))}\big) < \infty. \end{aligned} \tag{125}$$

Next we claim that

(a) it holds for all $k \in \{1, 2, \dots, n\}$, $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots, \delta_k) \in \mathbb{D}_k$ that

$$\sup_{\varepsilon, t \in (0,T]} \sup_{x \in H} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k) \in (H\backslash\{0\})^k} \left[\frac{t^{\iota_{\mathbb{N}}^{\boldsymbol{\delta},\alpha,\beta}} \, \|X_t^{\varepsilon,k,(x,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right] < \infty \tag{126}$$

and

(b) it holds for all $k \in \{1, 2, \dots, n\}$, $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots, \delta_k) \in \mathbb{D}_k$ with $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} < \infty$ that

$$\sup_{\varepsilon, t \in (0,T]} \sup_{\substack{x,y \in H,\\ x \neq y}} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k) \in (H\backslash\{0\})^k} \left[\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \, \|X_t^{\varepsilon,k,(x,\mathbf{u})} - X_t^{\varepsilon,k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right] < \infty. \tag{127}$$

We now prove item (a) and item (b) by induction on $k \in \{1, 2, \dots, n\}$. For the base case $k = 1$ we combine (115) and item (ii) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F_\varepsilon$, $B = B_\varepsilon$, $\alpha = \alpha$, $\beta = \beta$, $k = 1$, $p = p$, $\boldsymbol{\delta} = \delta$ for $\varepsilon \in (0, T]$, $\delta \in [0, 1/2)$, $p \in [2, \infty)$, in the notation of Theorem 2.1 in [2]) with (121)–(123) to obtain that for all $p \in [2, \infty)$, $\delta \in [0, 1/2)$ it holds that

$$\begin{aligned} &\sup_{\varepsilon, t \in (0,T]} \sup_{x \in H} \sup_{u \in H\backslash\{0\}} \left[\frac{t^{\delta} \, \|X_t^{\varepsilon,1,(x,u)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|u\|_{H_{-\delta}}} \right] \\ &\leq \chi_\delta \sup_{\varepsilon \in (0,T]} \Theta_p^{\delta}(|F_\varepsilon|_{C_b^1(H,H_{-\alpha})}, |B_\varepsilon|_{C_b^1(H,HS(U,H_{-\beta}))}) \\ &\leq \chi_\delta \sup_{\varepsilon \in (0,T]} \Upsilon_p^{\delta}(|F_\varepsilon|_{C_b^1(H,H_{-\alpha})}, |B_\varepsilon|_{C_b^1(H,HS(U,H_{-\beta}))}) \\ &\leq \chi_\delta \, \Upsilon_p^{\delta}(\chi_0 \, |F|_{C_b^1(H,H_{-\alpha})}, \chi_0 \, |B|_{C_b^1(H,HS(U,H_{-\beta}))}) < \infty. \end{aligned} \tag{128}$$

Moreover, combining (115) and item (iv) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F_\varepsilon$, $B = B_\varepsilon$, $\alpha = \alpha$, $\beta = \beta$, $k = 1$, $p = p$, $\boldsymbol{\delta} = \delta$ for $\varepsilon \in (0, T]$, $\delta \in [0, 1/2)$, $p \in \{r \in [2, \infty) \colon |F|_{\mathrm{Lip}^1(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^1(H,HS(U,H_{-\beta}))} < \infty\}$ in the notation of Theorem 2.1 in [2]) with (121)–(124) and (128) assures that for all $p \in [2, \infty)$, $\delta \in [0, 1/2)$ with $|F|_{\mathrm{Lip}^1(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^1(H,HS(U,H_{-\beta}))} < \infty$ it holds that

$$\begin{aligned} &\sup_{\varepsilon, t \in (0,T]} \sup_{\substack{x,y \in H,\\ x \neq y}} \sup_{u \in H\backslash\{0\}} \left[\frac{t^{\iota_{\mathbb{N}}^{(\delta,0),\alpha,\beta}} \, \|X_t^{\varepsilon,1,(x,u)} - X_t^{\varepsilon,1,(y,u)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H \, \|u\|_{H_{-\delta}}} \right] \\ &\leq |T \vee 1| \sup_{\varepsilon \in (0,T]} \Bigg\{ \Theta_p^{\iota_{\mathbb{N}}^{(\delta,0),\alpha,\beta}}\big(|F_\varepsilon|_{C_b^1(H,H_{-\alpha})}, |B_\varepsilon|_{C_b^1(H,HS(U,H_{-\beta}))}\big) \\ &\cdot \chi_0 \, \Theta_{2p}^0\big(|F_\varepsilon|_{C_b^1(H,H_{-\alpha})}, |B_\varepsilon|_{C_b^1(H,HS(U,H_{-\beta}))}\big) \sup_{t \in (0,T]} \sup_{x \in H} \sup_{u \in H\backslash\{0\}} \left[\frac{t^{\delta} \, \|X_t^{\varepsilon,1,(x,u)}\|_{\mathcal{L}^{2p}(\mathbb{P};H)}}{\|u\|_{H_{-\delta}}} \right] \\ &\cdot \left[\chi_\alpha \, \mathbb{B}\big(1-\alpha, 1-\delta\big) \|F_\varepsilon\|_{\mathrm{Lip}^1(H,H_{-\alpha})} + \chi_\beta \sqrt{\tfrac{p\,(p-1)}{2} \, \mathbb{B}\big(1-2\beta, 1-2\delta\big)} \, \|B_\varepsilon\|_{\mathrm{Lip}^1(H,HS(U,H_{-\beta}))} \right] \Bigg\} \end{aligned}$$

$$\begin{aligned}
&\leq \; |T \vee 1| \, |\chi_0|^2 \, \Upsilon_p^{\iota_{\mathbb{N}}^{(\delta,0),\alpha,\beta}} \big(\chi_0 \, |F|_{C_b^1(H,H_{-\alpha})}, \chi_0 \, |B|_{C_b^1(H,HS(U,H_{-\beta}))}\big) \\
&\cdot \Upsilon_{2p}^0\big(\chi_0 \, |F|_{C_b^1(H,H_{-\alpha})}, \chi_0 \, |B|_{C_b^1(H,HS(U,H_{-\beta}))}\big) \sup_{\varepsilon,t\in(0,T]} \sup_{x\in H} \sup_{u\in H\setminus\{0\}} \left[\frac{t^\delta \, \|X_t^{\varepsilon,1,(x,u)}\|_{\mathcal{L}^{2p}(\mathbb{P};H)}}{\|u\|_{H_{-\delta}}}\right] \\
&\cdot \left[\chi_\alpha \, \mathbb{B}\big(1-\alpha, 1-\delta\big) \|F\|_{\mathrm{Lip}^1(H,H_{-\alpha})} + \chi_\beta \sqrt{\tfrac{p\,(p-1)}{2}} \, \mathbb{B}\big(1-2\beta, 1-2\delta\big) \, \|B\|_{\mathrm{Lip}^1(H,HS(U,H_{-\beta}))}\right] \\
&< \; \infty.
\end{aligned} \tag{129}$$

This and (128) establish item (a) and item (b) in the base case $k = 1$. For the induction step $\{1, 2, \ldots, n-1\} \ni k \to k+1 \in \{2, 3, \ldots, n\}$ assume that there exists a natural number $k \in \{1, 2, \ldots, n-1\}$ such that item (a) and item (b) hold for all $l \in \{1, 2, \ldots, k\}$. Observe that (115), item (ii) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F_\varepsilon$, $B = B_\varepsilon$, $\alpha = \alpha$, $\beta = \beta$, $k = k+1$, $p = p$, $\boldsymbol{\delta} = \boldsymbol{\delta}$ for $\varepsilon \in (0, T]$, $\boldsymbol{\delta} \in \mathbb{D}_{k+1}$, $p \in [2, \infty)$ in the notation of Theorem 2.1 in [2]), the induction step, and (121)–(123) imply that for all $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1}) \in \mathbb{D}_{k+1}$ it holds that

$$\begin{aligned}
&\sup_{\varepsilon,t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_{k+1})\in(H\setminus\{0\})^{k+1}} \left[\frac{t^{\iota_{\mathbb{N}}^{\boldsymbol{\delta},\alpha,\beta}} \, \|X_t^{\varepsilon,k+1,(x,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^{k+1} \|u_i\|_{H_{-\delta_i}}}\right] \\
&\leq |T \vee 1|^{k+1} \sup_{\varepsilon\in(0,T]} \Bigg\{ \Theta_p^{\iota_{\mathbb{N}}^{\boldsymbol{\delta},\alpha,\beta}} \big(|F_\varepsilon|_{C_b^1(H,H_{-\alpha})}, |B_\varepsilon|_{C_b^1(H,HS(U,H_{-\beta}))}\big) \\
&\quad \cdot \Bigg[\chi_\alpha \, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^{k+1} \delta_i\big) \|F_\varepsilon\|_{C_b^{k+1}(H,H_{-\alpha})} \\
&\quad + \chi_\beta \sqrt{\tfrac{p\,(p-1)}{2}} \, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^{k+1} \delta_i\big) \, \|B_\varepsilon\|_{C_b^{k+1}(H,HS(U,H_{-\beta}))}\Bigg] \\
&\quad \cdot \sum_{\varpi\in\Pi_{k+1}^*} \prod_{I\in\varpi} \sup_{t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \, \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I} \|u_i\|_{H_{-\delta_i}}}\right]\Bigg\} \\
&\leq |T \vee 1|^{k+1} \, \chi_0 \, \Upsilon_p^{\iota_{\mathbb{N}}^{\boldsymbol{\delta},\alpha,\beta}} \big(\chi_0 \, |F|_{C_b^1(H,H_{-\alpha})}, \chi_0 \, |B|_{C_b^1(H,HS(U,H_{-\beta}))}\big) \\
&\quad \cdot \Bigg[\chi_\alpha \, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^{k+1} \delta_i\big) \|F\|_{C_b^{k+1}(H,H_{-\alpha})} \\
&\quad + \chi_\beta \sqrt{\tfrac{p\,(p-1)}{2}} \, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^{k+1} \delta_i\big) \, \|B\|_{C_b^{k+1}(H,HS(U,H_{-\beta}))}\Bigg] \\
&\quad \cdot \sum_{\varpi\in\Pi_{k+1}^*} \prod_{I\in\varpi} \sup_{\varepsilon,t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \, \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I} \|u_i\|_{H_{-\delta_i}}}\right] < \infty.
\end{aligned} \tag{130}$$

Furthermore, note that (115), item (iv) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F_\varepsilon$, $B = B_\varepsilon$, $\alpha = \alpha$, $\beta = \beta$, $k = k+1$, $p = p$, $\boldsymbol{\delta} = \boldsymbol{\delta}$ for $\varepsilon \in (0, T]$, $\boldsymbol{\delta} \in \mathbb{D}_{k+1}$, $p \in \{r \in [2, \infty) \colon |F|_{\mathrm{Lip}^{k+1}(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^{k+1}(H,HS(U,H_{-\beta}))} < \infty\}$ in the notation of Theorem 2.1 in [2]), (122), and (124) imply that for all $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1}) \in \mathbb{D}_{k+1}$ with $|F|_{\mathrm{Lip}^{k+1}(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^{k+1}(H,HS(U,H_{-\beta}))} < \infty$ it holds that

$$\sup_{\varepsilon,t\in(0,T]} \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_{k+1})\in(H\setminus\{0\})^{k+1}} \left[\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{\varepsilon,k+1,(x,\mathbf{u})} - X_t^{\varepsilon,k+1,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H \prod_{i=1}^{k+1} \|u_i\|_{H_{-\delta_i}}} \right]$$

$$\leq |T \vee 1|^{k+1} \sup_{\varepsilon\in(0,T]} \Bigg\{ \Theta_p^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \big(|F_\varepsilon|_{C_b^1(H,H_{-\alpha})}, |B_\varepsilon|_{C_b^1(H,HS(U,H_{-\beta}))}\big)$$

$$\cdot \Bigg(\chi_0\, \Theta^0_{(k+2)p} \big(|F_\varepsilon|_{C_b^1(H,H_{-\alpha})}, |B_\varepsilon|_{C_b^1(H,HS(U,H_{-\beta}))}\big)$$

$$\cdot \sum_{\varpi\in\Pi_{k+1}} \prod_{I\in\varpi} \sup_{t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \right]$$

$$+ \sum_{\varpi\in\Pi^*_{k+1}} \sum_{I\in\varpi} \sup_{t\in(0,T]} \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left[\frac{t^{\iota_{I\cup\{k+2\}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})} - X_t^{\varepsilon,\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\|x-y\|_H \prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \right]$$

$$\cdot \prod_{J\in\varpi\setminus\{I\}} \sup_{t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in J}\in(H\setminus\{0\})^{\#_J}} \left[\frac{t^{\iota_J^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\varepsilon,\#_J,(x,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in J} \|u_i\|_{H_{-\delta_i}}} \right] \Bigg)$$

$$\cdot \Bigg[\chi_\alpha\, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^{k+1} \delta_i\big) \|F_\varepsilon\|_{\mathrm{Lip}^{k+1}(H,H_{-\alpha})}$$

$$+ \chi_\beta \sqrt{\tfrac{p\,(p-1)}{2}\, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^{k+1} \delta_i\big)}\, \|B_\varepsilon\|_{\mathrm{Lip}^{k+1}(H,HS(U,H_{-\beta}))} \Bigg] \Bigg\}. \tag{131}$$

This, the induction hypothesis, (121), (123), (124), and (130) imply that for all $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1}) \in \mathbb{D}_{k+1}$ with $|F|_{\mathrm{Lip}^{k+1}(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^{k+1}(H,HS(U,H_{-\beta}))} < \infty$ it holds that

$$\sup_{\varepsilon,t\in(0,T]} \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_{k+1})\in(H\setminus\{0\})^{k+1}} \left[\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{\varepsilon,k+1,(x,\mathbf{u})} - X_t^{\varepsilon,k+1,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H \prod_{i=1}^{k+1} \|u_i\|_{H_{-\delta_i}}} \right]$$

$$\leq |T \vee 1|^{k+1}\, \chi_0\, \Upsilon_p^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \big(\chi_0\, |F|_{C_b^1(H,H_{-\alpha})}, \chi_0\, |B|_{C_b^1(H,HS(U,H_{-\beta}))}\big)$$

$$\cdot \Bigg(\chi_0\, \Upsilon^0_{(k+2)p} \big(\chi_0\, |F|_{C_b^1(H,H_{-\alpha})}, \chi_0\, |B|_{C_b^1(H,HS(U,H_{-\beta}))}\big)$$

$$\cdot \sum_{\varpi\in\Pi_{k+1}} \prod_{I\in\varpi} \sup_{\varepsilon,t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \right]$$

$$+ \sum_{\varpi\in\Pi^*_{k+1}} \sum_{I\in\varpi} \sup_{\varepsilon,t\in(0,T]} \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left[\frac{t^{\iota_{I\cup\{k+2\}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})} - X_t^{\varepsilon,\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\|x-y\|_H \prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \right]$$

$$
\begin{aligned}
&\cdot \prod_{J\in\varpi\setminus\{I\}} \sup_{\varepsilon,t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in J}\in(H\setminus\{0\})^{\#_J}} \left[\frac{t^{\iota_J^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\varepsilon,\#_J,(x,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in J}\|u_i\|_{H_{-\delta_i}}}\right]\Bigg) \\
&\cdot \Bigg[\chi_\alpha\, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^{k+1}\delta_i\big)\, \|F\|_{\mathrm{Lip}^{k+1}(H,H_{-\alpha})} \\
&+ \chi_\beta \sqrt{\tfrac{p\,(p-1)}{2}\, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^{k+1}\delta_i\big)}\, \|B\|_{\mathrm{Lip}^{k+1}(H,HS(U,H_{-\beta}))}\Bigg] < \infty.
\end{aligned} \tag{132}
$$

Combining (130) with (132) proves item (a) and item (b) in the case $k+1$. Induction hence establishes item (a) and item (b).

Next note that item (v) of Lemma 3.2 (with $n=n$, $\varphi=\varphi$, $F=F_\varepsilon$, $B=B_\varepsilon$, $X^{m,\mathbf{u}}=X^{\varepsilon,m,\mathbf{u}}$, $\phi=\phi_\varepsilon$, $k=k$, $\boldsymbol{\delta}=\boldsymbol{\delta}$, $\alpha=\alpha$, $\beta=\beta$ for $\boldsymbol{\delta}\in\mathbb{D}_k$, $k\in\{1,2,\ldots,n\}$, $\mathbf{u}\in H^{m+1}$, $m\in\{0,1,\ldots,n\}$, $\varepsilon\in(0,T]$ in the notation of Lemma 3.2), (122), item (a), and Jensen's inequality ensure that for all $k\in\{1,2,\ldots,n\}$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\ldots,\delta_k)\in\mathbb{D}_k$ it holds that

$$
\begin{aligned}
&\sup_{\varepsilon,t\in(0,T]} \sup_{v\in H} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \left[\frac{t^{\sum_{i=1}^k\delta_i} \left\|\big(\tfrac{\partial^k}{\partial x^k}\phi_\varepsilon\big)(t,v)\mathbf{u}\right\|_V}{\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right] \\
&\leq |T\vee 1|^{\lfloor k/2\rfloor \min\{1-\alpha,1/2-\beta\}}\, \|\varphi\|_{C_b^k(H,V)} \\
&\cdot \sum_{\varpi\in\Pi_k}\prod_{I\in\varpi} \sup_{\varepsilon,t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right] < \infty.
\end{aligned} \tag{133}
$$

This proves item (ii). It thus remains to prove item (iii). For this we combine item (viii) of Lemma 3.2 (with $n=n$, $\varphi=\varphi$, $F=F_\varepsilon$, $B=B_\varepsilon$, $X^{m,\mathbf{u}}=X^{\varepsilon,m,\mathbf{u}}$, $\phi=\phi_\varepsilon$, $k=k$, $\boldsymbol{\delta}=\boldsymbol{\delta}$, $\alpha=\alpha$, $\beta=\beta$ for $\boldsymbol{\delta}\in\mathbb{D}_k$, $k\in\{l\in\{1,2,\ldots,n\}\colon |F|_{\mathrm{Lip}^l(H,H_{-\alpha})}+|B|_{\mathrm{Lip}^l(H,HS(U,H_{-\beta}))}+|\varphi|_{\mathrm{Lip}^l(H,V)}<\infty\}$, $\mathbf{u}\in H^{m+1}$, $m\in\{0,1,\ldots,n\}$, $\varepsilon\in(0,T]$ in the notation of Lemma 3.2) with (122), (124), (125), item (a), item (b), and Jensen's inequality to obtain that for all $k\in\{1,2,\ldots,n\}$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\ldots,\delta_k)\in\mathbb{D}_k$ with $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})}+|B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))}+|\varphi|_{\mathrm{Lip}^k(H,V)}<\infty$ it holds that

$$
\begin{aligned}
&\sup_{\varepsilon,t\in(0,T]} \sup_{\substack{v,w\in H,\\ v\neq w}} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \left[\frac{t^{\sum_{i=1}^k\delta_i} \left\|\big[\big(\tfrac{\partial^k}{\partial x^k}\phi_\varepsilon\big)(t,v)-\big(\tfrac{\partial^k}{\partial x^k}\phi_\varepsilon\big)(t,w)\big]\mathbf{u}\right\|_V}{\|v-w\|_H\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right] \\
&\leq |T\vee 1|^{\lceil k/2\rceil \min\{1-\alpha,1/2-\beta\}}\, \|\varphi\|_{\mathrm{Lip}^k(H,V)} \\
&\cdot \sum_{\varpi\in\Pi_k}\Bigg\{ \sup_{\varepsilon,t\in(0,T]} \sup_{\substack{x,y\in H,\\ x\neq y}} \left[\frac{\|X_t^{\varepsilon,0,x}-X_t^{\varepsilon,0,y}\|_{\mathcal{L}^{\#_\varpi+1}(\mathbb{P};H)}}{\|x-y\|_H}\right] \\
&\cdot \prod_{I\in\varpi} \sup_{\varepsilon,t\in(0,T]} \sup_{x\in H} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \left[\frac{t^{\iota_I^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi+1}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right]
\end{aligned}
$$

$$+ \sum_{I \in \varpi} \sup_{\varepsilon, t \in (0,T]} \sup_{\substack{x,y \in H, \\ x \neq y}} \sup_{\mathbf{u}=(u_i)_{i \in I} \in (H \setminus \{0\})^{\#_I}} \left[\frac{t^{\iota_{I \cup \{k+1\}}^{(\boldsymbol{\delta},0),\alpha,\beta}} \|X_t^{\varepsilon,\#_I,(x,\mathbf{u})} - X_t^{\varepsilon,\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\|x-y\|_H \prod_{i \in I} \|u_i\|_{H_{-\delta_i}}} \right]$$

$$\cdot \prod_{J \in \varpi \setminus \{I\}} \sup_{\varepsilon, t \in (0,T]} \sup_{x \in H} \sup_{\mathbf{u}=(u_i)_{i \in J} \in (H \setminus \{0\})^{\#_J}} \left[\frac{t^{\iota_J^{\boldsymbol{\delta},\alpha,\beta}} \|X_t^{\varepsilon,\#_J,(x,\mathbf{u})}\|_{\mathcal{L}^{\#_\varpi}(\mathbb{P};H)}}{\prod_{i \in J} \|u_i\|_{H_{-\delta_i}}} \right] \Bigg\} < \infty. \tag{134}$$

This proves item (iii). The proof of Lemma 4.1 is thus completed. □

Corollary 4.2 *Assume the setting in* Section 1.2 *and let* $n \in \mathbb{N}$, $\alpha \in [0, 1)$, $\beta \in [0, 1/2)$, $F \in C_b^n(H, H_{-\alpha})$, $B \in C_b^n(H, HS(U, H_{-\beta}))$, $\varphi \in C_b^n(H, V)$. *Then*

(i) *it holds that there exist up-to-modifications unique* $(\mathcal{F}_t)_{t \in [0,T]}/\mathcal{B}(H)$*-predictable stochastic processes* $X^{\varepsilon,x} \colon [0, T] \times \Omega \to H$, $\varepsilon \in (0, T]$, $x \in H$, *which satisfy for all* $\varepsilon \in (0, T]$, $p \in (0, \infty)$, $x \in H$, $t \in [0, T]$ *that* $\sup_{s \in [0,T]} \mathbb{E}\big[\|X_s^{\varepsilon,x}\|_H^p\big] < \infty$ *and*

$$[X_t^{\varepsilon,x} - e^{tA}x]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s+\varepsilon)A} F(X_s^{\varepsilon,x})\, \boldsymbol{ds} + \int_0^t e^{(t-s+\varepsilon)A} B(X_s^{\varepsilon,x})\, dW_s, \tag{135}$$

(ii) *it holds that there exist unique functions* $\phi_\varepsilon \colon [0, T] \times H \to V$, $\varepsilon \in (0, T]$, *which satisfy for all* $\varepsilon \in (0, T]$, $t \in [0, T]$, $x \in H$ *that* $\phi_\varepsilon(t, x) = \mathbb{E}[\varphi(X_t^{\varepsilon,x})]$,

(iii) *it holds for all* $\varepsilon \in (0, T]$, $t \in [0, T]$ *that* $\big(H \ni x \mapsto \phi_\varepsilon(t, x) \in V\big) \in C_b^n(H, V)$,

(iv) *it holds for all* $k \in \{1, 2, \ldots, n\}$, $\delta_1, \delta_2, \ldots, \delta_k \in [0, 1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *that*

$$\sup_{\varepsilon, t \in (0,T]} \sup_{v \in H} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k) \in (H \setminus \{0\})^k} \left[\frac{t^{\sum_{i=1}^k \delta_i} \big\| \big(\frac{\partial^k}{\partial x^k} \phi_\varepsilon\big)(t, v)\mathbf{u} \big\|_V}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right] < \infty, \tag{136}$$

and

(v) *it holds for all* $k \in \{1, 2, \ldots, n\}$, $\delta_1, \delta_2, \ldots, \delta_k \in [0, 1/2)$ *with* $\sum_{i=1}^k \delta_i < 1/2$ *and* $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} + |\varphi|_{\mathrm{Lip}^k(H,V)} < \infty$ *that*

$$\sup_{\varepsilon, t \in (0,T]} \sup_{\substack{v,w \in H, \\ v \neq w}} \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k) \in (H \setminus \{0\})^k} \left[\frac{t^{\sum_{i=1}^k \delta_i} \big\| \big[\big(\frac{\partial^k}{\partial x^k} \phi_\varepsilon\big)(t, v) - \big(\frac{\partial^k}{\partial x^k} \phi_\varepsilon\big)(t, w) \big]\mathbf{u} \big\|_V}{\|v-w\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right] < \infty. \tag{137}$$

Proof Note that items (i) and (ii) follow immediately from item (i) of Theorem 2.1 in [2] (with $T = T$, $\eta = \eta$, $H = H$, $U = U$, $W = W$, $A = A$, $n = n$, $F = F$, $B = B$, $\alpha = 0$, $\beta = 0$ in the notation of Theorem 2.1 in [2]) and item (i) of Corollary 2.10 in [1]. Moreover, observe that items (iii)–(v) follow directly from items (i)–(iii) of Lemma 4.1. The proof of Corollary 4.2 is thus completed. □

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