Reformulation without f

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Lemma 0.0.1. *Let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\alpha_d \in \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, be infinitely often differentiable functions, let $u_d \in C^{1,2}\left([0,T] \times, \mathbb{R}^d, \mathbb{R}\right)$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that:*

$$\left(\frac{\partial}{\partial t} u_d\right)(t,x) + \left(\Delta_x u_d\right)(t,x) + \alpha_x\left(x\right) u_d\left(t,x\right) = 0 \tag{0.0.1}$$

Let $\mathcal{W}^d : [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$ be standard Brownian motions, and let $\mathcal{X}^{d,t,x} : [t,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $t \in [0,T]$, $s \in [t,T]$, $x \in \mathbb{R}^d$ we have $\mathbb{P}$-a.s. that:

$$\mathcal{X}^{d,t,x}_s = x + \int_s^t \sqrt{2} d\mathcal{W}^d_r \tag{0.0.2}$$

Then for all $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that:

$$u_d\left(t,x\right) + \mathbb{E}\left[\exp\left(\int_t^T \alpha_x\left(\mathcal{X}^{d,t,x}_r\right) dr\right) u_d\left(T, \mathcal{X}^{d,t,x}_T\right)\right] \tag{0.0.3}$$

Proof. Let $T \in [0, \infty)$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For all $d \in \mathbb{N}$, let $V \in C^{1,1}\left(\mathbb{R}^d \times [0,T], \mathbb{R}\right)$ be $V(x,t) = \alpha_d(x)$, let $\sigma_d : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ be given by $\sigma_d(x) = \text{diag}_d\left(\sqrt{2}\right)$, let $\mu_d : \mathbb{R}^d \to \mathbb{R}^d$ be given by $\mu_d(x) = \mathbb{0}_d$, and finally let $f(t,x) = 0$. By Feynman-Kac and substituting the above, the following expression:

$$\left(\frac{\partial}{\partial t} u_d\right)(t,x) + \frac{1}{2}\text{Trace}\left(\sigma(t,x)\left[\sigma(t,x)\right]^*\left(\text{Hess}_x(u_d)\left(t,x\right)\right) + \left\langle \mu(t,x), \left(\nabla_x u_d\right)(t,x)\right\rangle + V(t,x)u_d(t,x) + f(t,x) = 0 \tag{0.0.4}$$

is rendered:

$$\left(\frac{\partial}{\partial t} u_d\right)(t,x) + \left(\Delta_x u_d\right)(t,x) + \alpha_d(x) u_d(x) = 0 \tag{0.0.5}$$

Note then that Feyman-Kac sates that the solution to (0.0.4) can be written as:

$$u(t,x) = \mathbb{E}\left[\int_t^T e^{\int_t^r V(\mathcal{X}_t, \tau) d\tau} f(\mathcal{X}_r, r) dr + e^{-\int_t^T V(\mathcal{X}_\tau, \tau) d\tau} u(\mathcal{X}_T, T)\right] \tag{0.0.6}$$

Where $\mathcal{X}$ is an $(\Omega, \mathcal{F}, \mathbb{P})$-adapted stochastic process given by:

$$\mathcal{X}_t = x + \int_s^t \mu_d(\mathcal{X})\, dr + \int_s^t \sqrt{2} d\mathcal{W}_r^d \tag{0.0.7}$$

Note then that the substitutions then yield that the solution to (0.0.5) is given by:

$$u(t,x) = \mathbb{E}\left[\exp\left(\int_t^T \alpha_d(\mathcal{X})\, dr\right) u_d\left(T, \mathcal{X}_T^{d,t,x}\right)\right] \tag{0.0.8}$$

$\square$